Expressiveness of Positive Coalgebraic Logic

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Abstract

From the point of view of modal logic, coalgebraic logic over posets is the natural coalgebraic generalisation of positive modal logic. From the point of view of coalgebra, posets arise if one is interested in simulations as opposed to bisimulations. From a categorical point of view, one moves from ordinary categories to enriched categories. We show that the basic setup of coalgebraic logic extends to this more general setting and that every finitary functor on posets has a logic that is expressive, that is, has the Hennessy-Milner property.

Keywords: Coalgebra, Modal Logic, Poset

1 Introduction

We study the logic of coalgebras over posets and show that to any functor $T : \mathsf{Pos} \to \mathsf{Pos}$ one can associate a positive modal logic $\mathcal{L}_T$, that is, a modal logic without negation. Moreover, this logic has the Hennessy-Milner property (= is expressive) if T is finitary. For example, this extends to posets the familiar result that the modal logic **K** distinguishes non-bisimilar states of finitely

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branching Kripke models. We will also show expressiveness for some non-finitary functors such as the one giving rise to image-finite labelled transition systems with infinitely many labels.

As in the classical set-based situation the notion of (bi)similarity of interest is classified by the final T-coalgebra. But over Pos the final coalgebra carries a partial order, thus classifying similarity [34,40,17,21,29,9]. Accordingly, $\mathcal{L}_T$ will always be invariant under T-similarity and characterise it at least if T is finitary.

In the set-based situation, expressiveness follows if there is an *injection* mapping elements of the final coalgebra to their theories. Over Pos, if we want the logic not only to separate points but also to characterise the order on the final coalgebra, we need to consider order-reflecting injections (=*embeddings*). Consequently, whereas any set-functor T preserves injections (with non-empty domain), over Pos we need to explicitely require that T *preserves embeddings.*

Moreover, we want a strong expressiveness result stating that expressiveness can be achieved by *monotone* modal operators. As usual in the coalgebraic setting, we obtain modal operators from T via predicate liftings [31,36] and monotonicity of the modal operators in the usual sense coincides with monotonicity of the predicate liftings. As can be seen from (13), monotonicity of the predicate liftings requires the collection $[X, Y]$ of monotone maps from X to Y to be considered as a poset. Accordingly, also T will need to preserve the order between maps, that is, we need to require that T is *locally monotone.* Technically speaking this means that we are working in the setting of categories enriched over Pos.

To summarise then, from a technical point of view, we transfer to the setting of coalgebras enriched over Pos Schröder's theorem [36] stating that for any finitary set-functor the logic of all predicate liftings is expressive, which now becomes that for any finitary, locally-monotone, and embedding preserving poset-functor the logic of all monotone predicate liftings is expressive.

From a category theoretic point of view, one may ask whether instead of just treating Pos it would be more appropriate to immediatly treat locally presentable categories [6] in general. Whereas this seems entirely natural from the coalgebraic point of view, it is problematic from the logical point of view: In the spirit of Stone duality, both Set and Pos are in a dual adjunction with Boolean algebras (BA) and distributive lattices (DL), respectively. This allows us to systematically associate a logic $\mathcal{L}_T$ for T-coalgebras of any functor T on Set or Pos. The main idea here, going back to Domain Theory in Logical Form [1] and to the duality for modal algebras and Kripke frames [16], is to obtain the logic $\mathcal{L}_T$ from the functor $L : \mathsf{BA} \to \mathsf{BA}$ 'dual' to $T : \mathsf{Set} \to \mathsf{Set}$. That this is possible for arbitrary functors T on Set was shown in [26] and it is one contribution of this paper to show that this carries over from Set to Pos, as long as we are willing to work in the setting of categories enriched over Pos.

An important aspect of Stone duality is that, although we start with a general functor T, we obtain on the algebraic side a logic given concretely by a set of modal operators of finite arity and a set of equations. Furthermore, equational logic provides us with a proof system. To come back to the question of how to generalise beyond posets, it is not clear what then should replace distributive lattices and equational logic. We expect that future developments will take the lead from the observation that Pos itself is enriched over a two-element category of "truth-values", suggesting to replace Pos by a category of $\mathcal{V}$-categories [20] (rather than by a locally finitely presentable category), thus generalising to many-valued modal logics.

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2 Preliminaries

We review some basic material on the Stone duality approach to coalgebraic logic and on posets. More will be introduced later where the need arises.

2.1 Logical Connections

The basic ingredient of set-based coalgebraic logic is the adjunction

$$\begin{tikzcd} \mathsf{Set}^{op} \arrow[r, bend right, "Pred"'] \arrow[r, phantom, "\perp"] & \mathsf{BA} \arrow[l, bend right, "Stone"'] \end{tikzcd} \tag{1}$$

where *Pred* and *Stone* are the "predicate" and "Stone" functors, respectively. The functor *Pred* endows the powerset with the natural structure of a Boolean algebra and *Stone* takes the set of ultrafilters on a given Boolean algebra. Many nice properties of the above adjunction follow from the fact that it is given by a two-element set 2, that acts as a *schizophrenic object* in the sense of [33]. We will refer to the above adjunction as (an instance of) a *logical connection*.

Stone's representation theorem states that the unit $\eta_A : A \to \mathit{Pred}\,\mathit{Stone}\,A$ of the above adjunction is injective, which is a way of proving the completeness theorem of classical propositional logic.

By choosing the categories Set and BA we also have made a choice of over which category we will consider the coalgebras (here over Set), and where we will compute with the formulas of the relevant logic (here in BA).

Recall that, given a functor $T : \mathsf{Set} \to \mathsf{Set}$, a T*-coalgebra* (notation: (X, ξ) or just ξ) is a map $\xi : X \to TX$. A morphism $f : \xi \to \xi'$ is a map $f : X \to X'$ such that $Tf \cdot \xi = \xi' \cdot f$.

The rest of the set-based coalgebraic logic is therefore determined by a choice of a "behaviour" functor $T : \mathsf{Set} \to \mathsf{Set}$ and a functor $L : \mathsf{BA} \to \mathsf{BA}$ that captures the "logic" of coalgebras for T. The choice of T is made first and the functor L is subsequently computed to encode the modal operators and axioms describing T.

Thus, the full picture of set-based coalgebraic modal logic can be conveniently described by the following diagram [24]

$$\begin{tikzcd} \mathsf{Set}^{op} \arrow[loop left, "T^{op}"] \arrow[r, bend right, "Pred"'] \arrow[r, phantom, "\perp"] & \mathsf{BA} \arrow[l, bend right, "Stone"'] \arrow[loop right, "L"] \end{tikzcd} \tag{2}$$

The syntax and proof system of the induced modal logic are given by (a presentation) of the initial L-algebra[4] in BA and the semantics by a natural transformation

$$\delta_X : LPredX \to PredT^{op}X \tag{3}$$

Explicitly, δ associates to any coalgebra (X, ξ) the L-algebra $(PredX, Pred\xi \cdot \delta)$ and the map from the initial L-algebra to $PredX$ gives the semantics of the formulas of the logic.

Example 2.1 We recover Kripke frames and modal algebras by taking TX to be the powerset $\mathcal{P}X$ of X and LA to be the free Boolean algebra generated by $\{\Box a \mid a \in A\}$ modulo the equations stating that $\Box$ preserves finite meets. δ is defined by $\delta(\Box a) = \{Y \subseteq X \mid Y \subseteq a\}$.

That the category of T-coalgebras in the example above is isomorphic to the category of Kripke frames and bounded morphisms appears in [3], see also [35]. That the category of L-algebras is isomorphic to the category of modal algebras or Boolean algebras with operators is due to [2]. The generalisation of this classic correspondence [16] to general T is due to [26]. Let us also remark already that this example gives the logic of all predicate liftings of $\mathcal{P}$ since all predicate liftings can be obtained from $\Box$ and Boolean operations.

2.2 Posets

We are interested in coalgebras over the category Pos of posets and monotone maps. We denote by $V : \mathsf{Pos} \to \mathsf{Set}$ the forgetful functor and by $D : \mathsf{Set} \to \mathsf{Pos}$ its left-adjoint, which sends a set to the corresponding discrete poset. D has a further left-adjoint $C : \mathsf{Pos} \to \mathsf{Set}$ sending a poset to the set of its connected components. Consequently, D preserves limits and colimits. Note that $VD = \mathrm{Id}$.

Definition 2.2 *An* embedding $f : X \rightarrowtail Y$ *in* Pos *is a map that is monotone and order-reflecting, ie* $x \leq y \Leftrightarrow f(x) \leq f(y)$.

Proposition 2.3 *A morphism* $f : X \rightarrowtail Y$ *is an embedding in* Pos *if and only if it is a regular mono, that is, an equalizer.*

Notation. $\mathbb{2}$ denotes the linear order $0 < 1$. Given posets X,Y we write $[X, Y]$ for the poset of monotone maps, ordered pointwise.

Assumption. In order to be able to use the (enriched) Yoneda lemma, we assume that all functors $T : \mathsf{Pos} \to \mathsf{Pos}$ are *locally monotone*, that is, $f \leq g$ implies $Tf \leq Tg$.

2.3 Coalgebras over posets

Given a locally monotone $T : \mathsf{Pos} \to \mathsf{Pos}$, we will study the category $\mathsf{Coalg}(T)$ of T-*coalgebras*

$$\xi : X \to TX.$$

[4] An L-*algebra* (notation: (A, α) or just α) is an arrow $\alpha : LA \to A$ in BA. A morphism $f : \alpha \to \alpha'$ is an arrow $f : A \to A'$ in BA such that $f \cdot \alpha = \alpha' \cdot Lf$.

A *coalgebra morphism* $f : \xi \to \xi'$ is a monotone map $f : X \to X'$ such that $Tf \cdot \xi = \xi' \cdot f$ holds. We consider coalgebra homomorphisms to be ordered pointwise, i.e., $\mathsf{Coalg}(T)$ as *enriched* over Pos.

Coalgebras over posets have recently been studied by Levy [29]. Given a set-functor H and a so-called H-relator Γ, and following earlier work by, e.g., [38,17], he defines the notion of Γ-simulation between two H-coalgebras. Further he associates a functor $T : \mathsf{Pos} \to \mathsf{Pos}$ to Γ and shows that that the final T-coalgebra is fully abstract w.r.t. Γ-simulation. For our purposes we can summarise [29] as follows. Say that $R : X \twoheadrightarrow Y$ is a *monotone relation* from X to Y if $R \subseteq X \times Y$ and $R = \leq_X ; R ; \leq_Y$ where ; denotes relational composition. A monotone relation $R : X \twoheadrightarrow X'$ is a *simulation* from $\xi : X \to TX$ to $\xi' : X' \to TX'$ if

$$R \subseteq (\xi \times \xi')^{-1}((\mathsf{Rel}(T))(R)).$$

Here, $\mathsf{Rel}(T)$ is the relation lifting of T, that is, see [11],

$$\mathsf{Rel}(T)(R) = \{(a, a') \in TX \times TX' \mid \exists w \in TR \,.\, a = T\pi_X(w), a' = T\pi_{X'}(w)\}.$$

Alternatively, *similarity* can be defined via the final coalgebra: x is simulated by y if

$$!_\xi(x) \leq !_{\xi'}(x')$$

where ! denotes arrows into the final T-coalgebra.

The two definitions of similarity are equivalent under reasonable assumptions on the functor T by the Rutten-Worrell coinduction theorem [34, Thm 4.1], [40, Thm 5.10].

Example 2.4 We obtain syntax and (a slightly generalised) semantics of positive modal logic [15] by taking TX to be set of convex subsets of X and LA to be the free distributive lattice generated by $\{\Box a, \Diamond a \mid a \in A\}$ modulo the equations stating that $\Box$ preserves finite meets, $\Diamond$ preserves finite joins and the equations (1) of [15]. δ is defined by $\delta(\Box a)$ as in Example 2.1 and $\delta(\Diamond a) = \{Y \subseteq X \mid Y \cap a \neq \emptyset\}$.

In this example similarity agrees with bisimilarity due to the special nature of convex sets. The usual notions of similarity are obtained by taking upsets or downsets, see Example 3.4.

3 Functors on posets

The relationship between the modal logic $\mathbf{K}$ and positive modal logic [15] can be explained via the observation that the convex powerset functor is the extension of the powerset functor, see Definition 3.1.

Any finitary set functor H arises as a coequaliser

$$\coprod_{n,m<\omega} Hn \times \mathsf{Set}(n,m) \times X^m \rightrightarrows \coprod_{n<\omega} Hn \times X^n \longrightarrow HX \qquad (4)$$

where the upper map takes $(\sigma \in Hn, f : n \to m, v : m \to X)$ to $(\sigma, v \cdot f)$ and the lower map to $(Hf(\sigma), v)$. In more familiar notation, the coequaliser amounts to imposing the equations

$$\sigma(x_{f(1)}, \dots x_{f(n)}) = Hf(\sigma)(x_1, \dots x_m)$$

where n, m range over non-negative integers, f over maps $n \to m$, and σ over Hn.

Using the inclusion $D : \mathsf{Set} \to \mathsf{Pos}$, we can calculate this coequaliser not only for sets X but also for posets X, as follows.

Definition 3.1 *Let H be a finitary set functor. Define $\bar{H} : \mathsf{Pos} \to \mathsf{Pos}$ via the coequaliser in* Pos

$$\coprod_{n,m<\omega} DHn \times [Dn, Dm] \times [Dm, X] \rightrightarrows \coprod_{n<\omega} DHn \times [Dn, X] \longrightarrow \bar{H}X$$

Remark 3.2 (i) One reason for defining $\bar{H}$ via the coequaliser is that then $\bar{H}$ is locally monotone. The more immediate DHV (with $V : \mathsf{Pos} \to \mathsf{Set}$ the forgetful functor) is not locally monotone in general.

(ii) Note that $\bar{H}$ extends H in the sense that we have $\bar{H}D = DH$. The reason is that $D : \mathsf{Set} \to \mathsf{Pos}$ is a full co-reflective subcategory and, therefore, colimits of diagrams $\mathcal{C} \to \mathsf{Pos}$ factoring through D are already in Set, see the proof of Proposition 3.5.3 in Borceux [12, Vol 1].

(iii) D extends to a functor $\mathsf{Coalg}(H) \to \mathsf{Coalg}(\bar{H})$, due to $\bar{H}D = DH$.

Extensions of functors from Set to Pos are investigated in [9]. For example, we know that the final $\bar{H}$-coalgebra, if it exists, is discrete.

Example 3.3 (i) A *polynomial endofunctor* $H : \mathsf{Set} \to \mathsf{Set}$ is a functor given by

$$H(X) = \coprod_{n<\omega} \Sigma_n \times X^n.$$

The order on $\bar{H}(X, \leq)$ is the point-wise order induced by $(X, \leq)$.

(ii) The finite powerset functor $\mathcal{P}_\omega : \mathsf{Set} \to \mathsf{Set}$ extends to $\bar{\mathcal{P}_\omega} : \mathsf{Pos} \to \mathsf{Pos}$ mapping a poset X to the set of finitely generated convex subsets of X. The order on $\bar{\mathcal{P}_\omega}(X)$ is known as the Egli-Milner order, explicitly, for $A, B \in \bar{\mathcal{P}_\omega}(X)$ we have $A \leq B$ iff

$$\forall x \in A \,.\, \exists y \in B \,.\, x \leq y \;\wedge\; \forall y \in B \,.\, \exists x \in A \,.\, x \leq y,$$

With the exception of the first one, the following examples do not extend set-functors as they do not map discrete sets to discrete sets. Accordingly, interesting (non-symmetric) notions of similarity are obtained.

Example 3.4 (i) An example of a functor that does not preserve embeddings is the one which maps a poset to the discrete poset of its connected components.

(ii) $\mathcal{U}p_\omega : \mathsf{Pos} \to \mathsf{Pos}$ is the covariant functor which maps a poset to the set of all finitely generated up-sets ordered by reverse inclusion. Spelling out the definition of simulation from Section 2.3, we obtain that R is a simulation if $xRx' \;\Rightarrow\; \xi(x)\mathsf{Rel}(\mathcal{U}p_\omega)(R)\xi'(x')$, which is equivalent to

$$\forall y' \in \xi'(x') \,.\, \exists y \in \xi(x) \,.\, yRy'$$

(iii) $\mathcal{D}own_\omega : \mathsf{Pos} \to \mathsf{Pos}$ is the covariant functor which maps a poset to the set of all finitely generated down-sets ordered by inclusion. Here we have that R is a simulation if xRx' implies that

$$\forall y \in \xi(x) \,.\, \exists y' \in \xi'(x') \,.\, yRy'$$

(iv) Let A be a poset and $T : \mathsf{Pos} \to \mathsf{Pos}$, $TX = A \ltimes X$, where $\ltimes$ refers to the lexicographic ordering: $A \ltimes X$ has carrier $A \times X$ and the order is given by $(a, x) < (a', x') \;\Leftrightarrow\; (a < a' \vee (a = a' \wedge x < x'))$. In its second argument $\ltimes$ is functorial and locally monotone. Pavlović and Pratt [32] showed that the final $\mathbb{N} \ltimes \mathrm{Id}$-coalgebra is isomorphic to the non-negative real numbers.

(v) Consider $T : \mathsf{Pos} \to \mathsf{Pos}$, $TX = X^{\mathbb{2}}$, that is, X is mapped to the poset of pairs (x_1, x_2) with $x_1 \le x_2, x_1, x_2 \in X$.

(vi) Write $X \lhd X$ for the functor that makes two disjoint copies of X with everything on the left being smaller than anything on the right.

(vii) Allwein and Harrison [8] advertise the use of partially-ordered modalities. If A is an ordered set and $T : \mathsf{Pos} \to \mathsf{Pos}$ a functor, then $[A, T]$ is a functor which has A-indexed T-modalities. This generalises the approach of [8] to the situation where not only modalities, but also carriers of coalgebras may be partially ordered.

4 Logic for Coalgebras over Posets

Technically, in this paper, we replace the adjunction between Set and BA of diagram (1) by an adjunction

$$\begin{tikzcd} \mathsf{Pos}^{op} \arrow[r, bend right, "Pred"'] \arrow[r, phantom, "\perp"] & \mathsf{DL} \arrow[l, bend right, "Stone"'] \end{tikzcd} \tag{5}$$

between Pos and DL, the category of distributive lattices. The above adjunction is to be considered as an adjunction in the *enriched sense*. This means that both the predicate functor *Pred* and the Stone functor *Stone* are locally monotone and that there is an isomorphism

$$\mathsf{Pos}(X, \mathit{Stone}A) = \mathsf{Pos}^{op}(\mathit{Stone}A, X) \cong \mathsf{DL}(A, \mathit{Pred}X)$$

of *posets*, natural in X and A.

The predicate functor *Pred* assigns to a poset X the poset $[X, \mathbb{2}]$ of monotone maps $X \to \mathbb{2}$ endowed with the distributive lattice structure induced by $\mathbb{2}$. Observe that it means that the following diagram

$$\begin{tikzcd} \mathsf{Pos}^{op} \arrow[rr, "Pred"] \arrow[dr, "{[-,\mathbb{2}]}"'] & & \mathsf{DL} \arrow[dl, "U"] \\ & \mathsf{Pos} & \end{tikzcd} \tag{6}$$

commutes up to isomorphism, where U denotes the obvious (locally monotone!) forgetful functor from DL to Pos. Observe that, in elementary terms, $Pred X$ is the distributive lattice of upper-sets in X. Also note that there is an adjunction

$$F \dashv U : \mathsf{DL} \to \mathsf{Pos}.$$

The Stone functor $Stone : \mathsf{DL} \to \mathsf{Pos}^{op}$ assigns the poset $\mathsf{DL}(A, \mathbb{2})$ of prime filters on A to the distributive lattice A.

Notice that the adjunction of diagram (5) is built in the same way as the one of diagram (1), with $\mathbb{2}$ instead of 2. This “sameness” can be stated precisely by introducing schizophrenic objects and adjunctions they generate in the enriched setting. See [27] for the development of the theory along the lines of [33]. Let us just comment that basic ideas of [33] carry over to enrichment over Pos without any difficulties.

The whole picture of poset-based coalgebraic logic will therefore be given by the diagram

$$\begin{tikzcd}
\mathsf{Pos}^{op} \arrow[loop left, "T^{op}"] \arrow[r, bend right, "Pred"'] \arrow[r, phantom, "\perp"] & \mathsf{DL} \arrow[l, bend right, "Stone"'] \arrow[loop right, "L"] \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\
& \mathsf{Pos} \arrow[u, bend left, "F"]
\end{tikzcd} \tag{7}$$

As before the syntax and axioms of the logic will be given by a functor L and the semantics will be given by a natural transformation

$$\delta_X : LPred X \to Pred T^{op} X \tag{8}$$

Definition 4.1 *We call a pair* (L, δ) *as in* (7) *and* (8) *a* logic for T.

Remark 4.2 The adjunction $F \dashv U$ is important in order to be able to present the functor L concretely by operations and equations. As any finitary variety, DL is a completion inc : $\mathsf{DL}_0 \to \mathsf{DL}$ of the full subcategory DL_0 of finitely generated free algebras, see [7] for details. Consequently, any functor $L' : \mathsf{DL}_0 \to \mathsf{DL}$ can be extended continuously to a functor $L : \mathsf{DL} \to \mathsf{DL}$. Technically, this can be expressed by saying that L is the left-Kan extension [28] of L' along inc. Conversely, if a functor L arises as such a left-Kan extension, we say that L *is determined by finitely generated free algebras*, i.e., in the terminology of [39], L is *finitely based* w.r.t. $F \dashv U$. The important fact for us is that such a functor can be presented by operations and equations [26]. We do not have the space for a full account on this, but we will see more details in Section 4.2.

We think of δ as the one-step semantics of the logic and may write for $y \in TX$ and $b \in LPred X$

$$y \Vdash b \;\Leftrightarrow\; y \in \delta(b) \tag{9}$$

To go from the one-step semantics to the ‘global’ semantics, we have to iterate the one-step logic-constructor L and form the initial L-algebra as the colimit of the initial L-sequence, see, e.g., [24]. This colimit exists if L is finitary, which will be the case in the examples we will look at later. For now we assume that the initial L-algebra $L\mathcal{L} \to \mathcal{L}$ exists and we consider $\mathcal{L}$ as the set of formulas of the logic. The semantics of a formula $\varphi \in \mathcal{L}$ w.r.t. a coalgebra $\xi : X \to TX$ is then given by the unique L-algebra morphism $[\![\cdot]\!]_\xi$

$$\begin{tikzcd} L\mathcal{L} \arrow[rr] \arrow[d, "L[\![\cdot]\!]_\xi"'] & & \mathcal{L} \arrow[d, "[\![\cdot]\!]_\xi"] \\ LPredX \arrow[r, "\delta_X"] & PredTX \arrow[r, "Pred\xi"] & PredX \end{tikzcd} \tag{10}$$

If L has a presentation by operations and equation as in Remark 4.2, a formula can be represented as $\boxdot(\varphi_1, \dots \varphi_n)$ where $\boxdot$ is an n-ary 'modal' operation symbol and $\varphi_i \in \mathcal{L}$. Then (10) can be written as the inductive clause

$$x \in [\![\boxdot(\varphi_1, \dots \varphi_n)]\!]_\xi \;\Leftrightarrow\; \xi(x) \in \delta_X(\boxdot([\![\varphi_1]\!]_\xi, \dots [\![\varphi_n]\!]_\xi))$$

4.1 Expressiveness

We discuss the expressiveness of coalgebraic logics following Klin [23, Theorem 4.2] (see [18, Theorem 4]). Let us say that a logic for T-coalgebras is *expressive* if all elements x, y of the final T-coalgebra *can be separated by a formula*, or, more precisely, if we have

$$x \not\leq y \;\Rightarrow\; \exists \varphi \in \mathcal{L} \,.\, x \Vdash \varphi \;\&\; y \nVdash \varphi \tag{11}$$

This means that the logic not only separates elements, but also characterises the order, namely, $x \leq y$ iff $\forall \varphi \in \mathcal{L} \,.\, x \Vdash \varphi \Rightarrow y \Vdash \varphi$.

The formal treatment will follow an idea analogous to the one introduced by Pattinson [31] for completeness arguments: We first define what it means to be *one-step* expressive and then show that one-step expressiveness extends to expressiveness. One advantage of this one-step approach is that it often gives rise to modularity [13,37]: If we are interested in an inductively defined class of functors and we can show that all constructions preserve a certain property P one-step-wise, then it follows that all functors in that class have property P.

Coming back to one-step expressiveness, first note that the natural transformation δ induces its *mate* $\tau : T^{op}Stone \to StoneL$ given explicitly by the pasting [5]

$$\begin{tikzcd} & \mathsf{Pos}^{op} \arrow[rr, "T^{op}"] \arrow[dr, "Pred"] & & \mathsf{Pos}^{op} \arrow[r, equal] \arrow[dr, "Pred"'] & \mathsf{Pos}^{op} \\ \mathsf{DL} \arrow[ur, "Stone"] \arrow[rr, equal] & \uparrow\eta & \mathsf{DL} \arrow[rr, "L"'] & \uparrow\delta & \mathsf{DL} \arrow[u, phantom, "\uparrow\varepsilon"] \arrow[ur, "Stone"'] \end{tikzcd} \tag{12}$$

where η and ε are the unit and the counit of the adjunction $Stone \dashv Pred$, see (7). Explicitly, using the notation from (9), we have

$$\begin{aligned} \tau_A : T^{op}StoneA &\to StoneLA \\ y &\mapsto \{b \in LA \mid y \Vdash\!\!\Vdash L(\eta)(b).\} \end{aligned}$$

[5] That is, in Pos^{op} we have that τ is given by $\varepsilon T^{op}Stone \cdot Stone\delta Stone \cdot StoneL\eta$.

Thus, if τ_A is injective, then all elements of $T^{op}StoneA$ can be separated by some 'one-step formula' $b \in LA$ and if, moreover, τ_A is an embedding then the logic even characterises the order on $T^{op}StoneA$. Thus we see that τ being an embedding is the one-step version of (11).

Alternatively, one could try to define one-step expressiveness by the surjectivity of δ, but this is too strong: It means that every subset is the extension of some formula.

Definition 4.3 *We call* (L, δ) *expressive if* (11) *holds and one-step expressive if the mate* τ *of* δ *is an embedding.*

Remark 4.4 These definitions work over Set as well as over Pos. Note that basic modal logic is not one-step expressive for $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$. Indeed, if $\alpha : LA \to A$ is the Lindenbaum-algebra of the logic, then α is an isomorphism and we have $\tau_A : \mathcal{P}^{\mathrm{op}}StoneA \to StoneLA \cong StoneA$, which is not injective. On the other hand, basic modal logic is one-step expressive for the finite powerset $\mathcal{P}_\omega$.

Theorem 4.5 *Let* $T : \mathsf{Pos} \to \mathsf{Pos}$ *be finitary and embedding-preserving and let* (L, δ) *be a logic for* T*. If the logic is one-step expressive, then it is expressive.*

Proof. We have the following diagram in Pos:

$$\begin{tikzcd}
Stone\mathbb{2} = 1 \arrow[r] & StoneL\mathbb{2} \arrow[r] & StoneL^2\mathbb{2} \arrow[r] & \ldots \arrow[r] & StoneL^k\mathbb{2} \arrow[r] & {} \\
1 \arrow[u, "\theta_0"] \arrow[r] & T1 \arrow[u, "\theta_1"] \arrow[r] & T^2 1 \arrow[u, "\theta_2"] \arrow[r] & \ldots \arrow[r] & T^k 1 \arrow[u, "\theta_k"] \arrow[r] & {}
\end{tikzcd}$$

We define θ_0 to be the identity on 1 and $\theta_{k+1} = \tau_{L^k\mathbb{2}} \cdot T(\theta_k)$. We need an auxiliary claim: There is an ordinal α such that $T^\alpha 1$ is the final coalgebra and $T^\alpha 1 \to T^\omega 1$ is an embedding. Indeed, generalising a well-known result of Worrell [41], Adámek [4] shows in the proof of his Theorem 4.6 the following: (1a) $T^{\omega+1}1 \to T^\omega 1$ is an embedding; (1b) this property is preserved at successor ordinals; (1c) this property is preserved at limit ordinals. [6] Consequently, beyond ω, the final sequence consists only of embeddings and thus must converge to the final coalgebra.

Now suppose $x \not\leq y$ in the final coalgebra $\zeta : T^\alpha 1 \to T^{\alpha+1}1$. Due to the claim, we have $x \not\leq y$ in $T^\omega 1$. But this means that there is $n < \omega$ such that $x \not\leq y$ in $T^n 1$. Since T preserves embeddings, θ_n is an embedding, hence $\theta_n(x) \not\subseteq \theta_n(y)$. Thus there must be $\varphi \in \theta_n(x)$ with $\varphi \notin \theta_n(y)$. It is routine, if lengthy, to verify that this implies that $x \in [\![\varphi]\!]_\zeta$ and $y \notin [\![\varphi]\!]_\zeta$. $\square$

Remark 4.6 The proof can be seen as a final sequence version of the proof of Klin [23, Theorem4.2], which in turn can be seen as a category theoretic analysis of Schröder [36].

[6] This part of the proof of Theorem 4.6 in [4] does not need its assumption that T preserves epis; and the argument for (1c) is not specific to the ordinal 2ω.

A benefit of the final sequence approach is that we also get a result for functors that are not necessarily finitary. Adapting [25], we say about two states x, x' of two coalgebras $\xi : X \to TX, \xi' : X' \to TX'$ that x is *ω-simulated* by x' if $!_n(x) \leq !_n(x')$ for all $n < \omega$, where $!_n(x) : X \to T^n 1$ is the map induced by ξ. A logic is called *ω-expressive* if whenever x is not ω-simulated by x', then $\exists \varphi \in \mathcal{L} \,.\, x \Vdash \varphi \;\&\; y \nVdash \varphi$. The proof of the following corollary then repeats the last three sentences of the proof of the theorem.

Corollary 4.7 *Let T be a (not necessarily finitary) endofunctor on* Pos *preserving embeddings and let (L, δ) be a one-step expressive logic for T.*[7] *Then (L, δ) is ω-expressive.*

4.2 Predicate Liftings

As in the Set-based case L and δ can be described more explicitly by predicate liftings, introduced by Pattinson [31]. To transport this approach to our setting, we only need to generalise the n-th power (n is a finite set) of a set S to the "n-th power of a poset S", where n now a finite poset. It turns out that the universal property (natural in X)

$$\mathsf{Set}(X, S^n) \cong \mathsf{Set}(n, [X, S])$$

of the n-fold power of a set S can be taken verbatim to define the desired (enriched) notion for posets.

Definition 4.8 *Suppose S is a poset and n is a finite poset. The* n-fold cotensor *of S is a poset $n \pitchfork S$ together with an isomorphism*

$$\mathsf{Pos}(X, n \pitchfork S) \cong \mathsf{Pos}(n, [X, S])$$

natural in X, where $[X, Y]$ denotes the poset of monotone maps from X to Y.

It is easy to verify that $n \pitchfork S$ is the poset $[n, S]$ of all monotone maps from n to S, ordered pointwise. The reason to introduce the $\pitchfork$ notation is that it carries over if one wants to replace Pos in (5) by, for example, Priestley spaces [14] (which are dually equivalent to DL): then X, S are Priestley spaces but n remains a poset, so we could not write $[n, S]$ anymore.

Definition 4.9 *An n-ary* predicate lifting *for a functor T is a natural transformation λ given by components*

$$\lambda_X : [X, n \pitchfork 2] \to [TX, 2] \tag{13}$$

where n can be any finite poset.

As opposed to the Set-based case, all predicate liftings are monotone since each λ_X is an arrow in Pos and hence a monotone map.

[7] One-step expressiveness can be weakened. It is enough to require that the τ_A are embeddings for those $A = L^k 2$ appearing in the final sequence construction. An example is given by the (not-necessarily finitely generated) convex subsets functor.

Remark 4.10 It follows from the (enriched) Yoneda lemma that the poset of all predicate liftings is order-isomorphic to

$$[T(n \pitchfork 2), 2]. \tag{14}$$

Recall also that, for every finite poset n, the poset $[T(n \pitchfork 2), 2]$ is an underlying poset of a distributive lattice that we denote by Λn (where Λ should remind of lifting). The nature of the adjunction (5) and diagram (6) allows us to infer the isomorphism

$$\Lambda n \cong \mathit{Pred} T^{op} \mathit{Stone} F n \tag{15}$$

where Fn denotes the free DL over a finite poset n. The assignment $n \mapsto \Lambda n$ can be viewed as a finitary signature in the enriched sense: arities are finite posets and n-ary operations (i.e., n-ary predicate liftings) form a distributive lattice.

The following remark explains how predicate liftings fit into diagram (7). Recall that $U : \mathsf{DL} \to \mathsf{Pos}$ denotes the forgetful functor.

Remark 4.11 Predicate liftings induce a functor $L : \mathsf{DL} \to \mathsf{DL}$. The idea is that LA is the distributive lattice of "modal formulas of depth one, labelled in elements of a distributive lattice A". Hence L is the polynomial functor corresponding to the above signature Λ and the precise formula is given by

$$LA = \coprod_{n \text{ finite poset}} [n, UA] \otimes \Lambda n \tag{16}$$

where by $\otimes$ we denote the $[n, UA]$-fold *tensor* of the distributive lattice Λn. In general, P-fold tensor of a distributive lattice A is a distributive lattice $P \otimes A$ together with an isomorphism

$$\mathsf{DL}(P \otimes A, B) \cong \mathsf{Pos}(P, \mathsf{DL}(A, B))$$

natural in B. Since (16) is a left Kan extension formula (in the appropriate enriched sense), L comes equipped for canonical reasons with a natural transformation given by:

$$\delta_X : L\mathit{Pred}X \to \mathit{Pred}T^{op}X \tag{17}$$

$$\lambda(a_1, \ldots, a_n) \mapsto \lambda \cdot Ta' : TX \to T(n \pitchfork 2) \to 2, \tag{18}$$

where $a' : X \to n \pitchfork 2$ is the transpose of $a : n \to U\mathit{Pred}X = [X, 2]$, see Definition 4.8.

Remark 4.12 Instead of taking all predicate liftings as in (15), we can consider any collection $\Lambda'_n \subseteq U\Lambda n$ of predicate liftings. It only means that in (16) we replace Λn by the DL freely generated from Λ'_n. The corresponding logic (L, δ) is then still given by (18) where now $\lambda \in \Lambda'_n$.

Proposition 4.13 *The functor L preserves surjective homomorphisms.*

Proof. As L is determined by its values on finitely generated free algebras, see Remark 4.2, it preserves sifted colimits and, therefore, reflexive coequalisers [7]. But every surjective homomorphism is the reflexive coequaliser of its kernel pair. $\square$

4.3 The Logic of all Predicate Liftings is Expressive

The next theorem shows that the logic of all predicate liftings, or also the logic of all predicate liftings with discrete arities, is expressive. The key observations are contained in the following lemma.

Lemma 4.14 *Every finite poset X can be embedded into a poset $n \pitchfork \mathbb{2}$ for some finite discrete poset n.*

Proof. Write n for the discrete poset of elements of X. The embedding is given by $X \to [X^{\mathrm{op}}, \mathbb{2}] \to [n, \mathbb{2}]$ where the first map takes principal down-sets and the second is the embedding from monotone maps $X^{\mathrm{op}} \to \mathbb{2}$ into all not-necessarily monotone maps. $\square$

The next lemma requires embeddings as opposed to only injections. It is also the place where it comes in that DL has finite meets and joins. Finally, we note for future generalisations that the lemma makes use of the following: (i) every poset is a filtered colimit of finite posets where we can take the cone to consist of embeddings (reg monos); (ii) if X is finite and $X \to \mathit{Stone}A$ is reg mono then the transpose $A \to \mathit{Pred}X$ is reg epi; (iii) L preserves reg epis.

Lemma 4.15 *Consider a finite poset X and an embedding $c : X \to \mathit{Stone}A$. Denote by $c^\sharp : A \to \mathit{Pred}X$ the transpose of c. Then $\mathit{Stone}Lc^\sharp : \mathit{Stone}L\mathit{Pred}X \to \mathit{Stone}LA$ is an embedding in Pos.*

Proof. We first prove that $c^\sharp$ is a surjection. To this end, it is convenient to identify $\mathit{Pred}X$ with the set of upsets of X (ordered by inclusion), $\mathit{Stone}A$ with the set of prime filters on A (ordered by inclusion), and to abbreviate $a \in c(x)$ by $x \Vdash a$. In this notation $c(x) = \{a \in A \mid x \Vdash a\}$ and $c^\sharp(a) = \{x \in X \mid x \Vdash a\}$. Moreover, that c is an embedding means that $x \not\leq y$ iff there is $a_{xy} \in A$ such that $x \Vdash a_{xy}$ and $y \nVdash a_{xy}$. Now consider a 'principal' upset $\uparrow x \in \mathit{Pred}X$. Since X is finite we find $a_x = \bigwedge\{a_{xy} \mid y \in X \text{ and } x \not\leq y\}$ in A. It follows $c^\sharp(a_x) = \uparrow x$. Since every upset in $\mathit{Pred}X$ is a finite join of principal upsets, $c^\sharp$ is onto.

Now that $c^\sharp$ is a surjection, by Proposition 4.13, we have that $Lc^\sharp$ is a surjection as well. But in all equational classes of algebras all epimorphisms are regular, so $Lc^\sharp$ is a regular epi in DL. Finally, $\mathit{Stone}Lc^\sharp$ is a regular epi in Pos^{op} since it is an image of a regular epi under a left adjoint. $\square$

Theorem 4.16 *If $T : \mathsf{Pos} \to \mathsf{Pos}$ is finitary, locally monotone, and embedding-preserving, and Λ consists of all predicate liftings with finite arities, then τ is an embedding.*

Proof. (For reasons of type setting, we write $S = \mathit{Stone}$ and $P = \mathit{Pred}$ inside this proof.) By the definition of τ, we need to show that the composite

$$SL\varepsilon_A \cdot S\delta_{SA} \cdot \eta_{TSA} : TSA \to SPTSA \to SLPSA \to SLA$$

is an embedding. Consider $c : X \to SA$ and its transpose $c^\sharp : A \to PX$. Now

$$
\begin{tikzcd}
TSA \arrow[r, "\eta_{TSA}"] & SPTSA \arrow[r, "S\delta_{SA}"] & SLPSA \arrow[r, "SL\varepsilon_A"] & SLA \\
TX \arrow[u, "Tc"] \arrow[r, "\eta_{TX}"] & SPTX \arrow[r, "S\delta_X"] & SLPX \arrow[ur, "SLc^\sharp"'] &
\end{tikzcd}
\tag{19}
$$

commutes because of the naturality of η and δ. Further, Pos being a locally finitely presentable category [6], SA is a filtered colimit $c_i : X_i \to SA$ where we can take the X_i to be finite and the c_i to be embeddings. Therefore, since T preserves filtered colimits, we only need that the lower composite of (19)

$$SLc^\sharp \cdot S\delta_X \cdot \eta_{TX} : TX \to SPTX \to SLPX \to SLA$$

is an embedding for finite X and embeddings $c : X \to SA$. By Lemma 4.15 we need to show that

$$\alpha = S\delta_X \cdot \eta_{TX} : TX \to SPTX \to SLPX$$

is an embedding. According to (18), α maps a point $t \in TX$ together with a formula $\lambda(a)$ to its truth value via

$$\lambda \cdot Ta : TX \to T(n \pitchfork \mathbb{2}) \to \mathbb{2}. \tag{20}$$

To show that α is order-reflecting, consider $t_1, t_2 \in TX$, $t_1 \not\leq t_2$. Due to Lemma 4.14 and T preserving embeddings, there is $a : X \to n \pitchfork \mathbb{2}$ such that $Ta(t_1) \not\leq Ta(t_2)$. Define $\lambda : T(n \pitchfork \mathbb{2}) \to \mathbb{2}$ as $\lambda(x) = 1 \Leftrightarrow Ta(t_1) \leq x$, which is monotone. We now have $\lambda(Ta(t_1)) \not\leq \lambda(Ta(t_2))$, hence α is order-reflecting. □

Corollary 4.17 *Let $T : \mathsf{Pos} \to \mathsf{Pos}$ be finitary, locally monotone, and embedding-preserving. The logic of all predicate liftings (with discrete arities) is expressive.*

Remark 4.18 This theorem is not a consequence of [23, Thm 4.4] since we need strong monos instead of monos and also because we want that discrete arities suffice for expressiveness. For a precise comparison of the two theorems, we give below a common generalisation of both, but the only instances we know are Theorem 4.16 and [23, Thm 4.4].

Theorem: Let $\mathcal{X}$ be a locally finitely presentable category [6], $F \dashv U : \mathcal{A} \to \mathsf{Set}$, $\mathcal{A}_0 \hookrightarrow \mathcal{A}$ a full, dense subcategory, and $P \dashv S : \mathcal{A}^{\mathrm{op}} \to \mathcal{X}$. Let $LA = PTSA$ for $A \in \mathcal{A}_0$ and extend via colimits to $L : \mathcal{A} \to \mathcal{A}$. Let $(\mathcal{E}, \mathcal{M})$ be either the (StrongEpi,Mono) or a (Epi,RegMono) factorisation system [6, 1.61] on $\mathcal{X}$. Moreover we require: T preserves arrows in $\mathcal{M}$; the unit $X \to SPX$ is pointwise in $\mathcal{M}$; (*) given X finitely presentable, $X \to \bar{X}$ in $\mathcal{E}$, and $m : \bar{X} \to SA$ in $\mathcal{M}$, there is $A' \in \mathcal{A}$ and $\varphi : A' \to A$ such that $S\varphi \cdot m$ in $\mathcal{M}$. Then the mate $\tau : TS \to ST$ of $\delta : LP \to PT$ is in $\mathcal{M}$.

The technical condition (*) can be established in the case of (Epi,StrongMono)-factorisation if $\mathcal{X}$ is, as in [23, Thm 4.4], strongly locally finitely presentable. In our case, (*) follows from Lemma 4.15.

An interesting example of a non-finitary functor that has an expressive logic is $\mathcal{P}_\omega^{Act}$ where Act is an infinite set of 'actions' and $\mathcal{P}_\omega$ is finite powerset. Its coalgebras are image-finite transition systems. We can extend our results to these in the manner of [41].

Theorem 4.19 *If T_i is a family of finitary, locally monotone functors preserving embeddings, then $\prod_i T_i$ has an expressive logic.*

Proof. By Theorem 4.16 each T_i has a logic (L_i, δ_i) with the mates $\tau_i : T_iS \to SL_i$ being embeddings. Hence $\tau : \prod_i T_iS \to \prod_i SL_i \cong S \coprod_i L_i$ is an embedding and therefore $(\coprod L_i, \delta)$ with now δ being the mate of τ is one-step expressive. It follows from Corollary 4.7 that $\coprod_i L_i$ is ω-expressive and from (the poset-version of) [41, Theorem 13] that the final T-coalgebra is embedded in the ω-limit of the final sequence, hence ω-expressiveness implying expressiveness. $\Box$

Remark 4.20 The same proofs, with Pos replaced by Set also give proofs of the expressivity of coalgebraic logic over Set. The set-analogue of Theorem 4.19 is also of interest. We also strengthen the results of [36,22] in that monotone predicate liftings with negation-free propositional logic are expressive for finitary set-functors.

4.4 Separating Sets of Predicate Liftings

In concrete examples, we are interested in generating logics from small sets Λ of predicate liftings. Going back to Remark 4.12 and (18) and (20), we see that for expressivity, it is enough to require that the set of predicate liftings is separating. We adapt this notion from [31,36] to the ordered setting.

Definition 4.21 *A collection Λ' of predicate liftings is* separating *if the family*

$$(\widehat{\lambda}_X : TX \to [[X, n \pitchfork 2], 2])_{\lambda \in \Lambda n}$$

is jointly order-reflecting, that is, for all finite X and all $t_1, t_2 \in TX$ with $t_1 \not\leq t_2$ there are a finite poset n and $a : X \to n \pitchfork 2$ and $\lambda \in \Lambda'_n$ such that $\lambda \cdot Ta(t_1) = 1$ and $\lambda \cdot Ta(t_2) = 0$.

An inspection of the proof of Theorem 4.16 shows

Proposition 4.22 *A collection Λ' of predicate liftings is separating only if the mate τ of δ is an embedding.*

Corollary 4.23 *If Λ' is separating then the logic given by Λ' is expressive.*

Let us look at a couple of the Examples 3.4

Example 4.24 (i) For $T = \mathcal{U}p_\omega : \mathsf{Pos} \to \mathsf{Pos}$ we take one unary predicate lifting: LA is generated by $\Box a, a \in A$. (8) is given by

$$\Box a \quad \mapsto \quad \texttt{lambda}\ b.\texttt{if}\ b \subseteq a\ \texttt{then}\ 1\ \texttt{else}\ 0 : \mathcal{U}pX \to 2$$

which we can also read as defining a predicate lifting in the form of (13) (with $a \in \mathsf{Pos}(X, 2)$ and $b \in \mathcal{U}p(X)$). In the form of (14) it is a function $\mathcal{U}p(2) \to 2$ mapping $\{0, 1\}$ to 0 and $\{1\}$ and $\emptyset$ to 1.

(ii) For the functor $X \mapsto X \triangleleft X$ we take two unary modal operators $[l]$ and $[r]$ and, in addition to usual axioms for disjoint union saying that $[l]$ and $[r]$ preserve the DL-operations, we also have $[l]a \leq [r]a$, reflecting that formulas denote up-sets.

5 Conclusion

We have developed the predicate lifting approach to coalgebras over posets. Let us note that this also includes coalgebras over the category Pre of preorders: The adjunction $\mathsf{Pre}^{\mathrm{op}} \rightleftarrows \mathsf{Pos}^{\mathrm{op}} \rightleftarrows \mathsf{DL}$ retains all the necessary properties from (5). The purpose of this observation is the importance of Pre as a base category for coalgebras in the study of simulation: Since two-way simulation is in general different from bisimulation, one needs to work with preorders if one wants to classify bisimulation and simulation in the same final coalgebra. It should be of interest to use this to study in a systematic way logics of simulation and how they relate to logics of bisimulation.

The logic of all monotone predicate liftings is also of interest for set-functors, for example if one wants to extend coalgebraic logics by fixed points. The logic of all monotone predicate liftings of a functor $H : \mathsf{Set} \to \mathsf{Set}$ can be investigated via the logic of all (necessarily monotone) predicate liftings $\bar{H} : \mathsf{Pos} \to \mathsf{Pos}$ as in Definition 3.1, leading to a systematic investigation into the relationship between the BA-logic for T-coalgebras and their positive DL-fragments [10].

Further work on which we embarked already includes an account of Moss's [30] cover modality ∇ over Pos [11]. Investigations into many-valued logics based on replacing $\mathbb{2}$ by a commutative quantale are ongoing.

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A Proof of Remark 4.18

Theorem: Let $\mathcal{X}$ be a locally finitely presentable category [6], $F \dashv U : \mathcal{A} \to \mathsf{Set}$, $\mathcal{A}_0 \hookrightarrow \mathcal{A}$ a full, dense subcategory, and $P \dashv S : \mathcal{A}^{\mathrm{op}} \to \mathcal{X}$. Let $LA = PTSA$ for $A \in \mathcal{A}_0$ and extend via colimits to $L : \mathcal{A} \to \mathcal{A}$. Let $(\mathcal{E}, \mathcal{M})$ be either the (StrongEpi,Mono) or a (Epi,RegMono) factorisation system [6, 1.61] on $\mathcal{X}$. Moreover we require: T preserves arrows in $\mathcal{M}$; the unit $X \to SPX$ is pointwise in $\mathcal{M}$; (*) given X finitely presentable, $X \to \bar{X}$ in $\mathcal{E}$, and $m : \bar{X} \to$

SA in $\mathcal{M}$, there is $A' \in \mathcal{A}$ and $\varphi : A' \to A$ such that $S\varphi \cdot m$ in $\mathcal{M}$. Then the mate $\tau : TS \to ST$ of $\delta : LP \to PT$ is in $\mathcal{M}$.

Remark. $\mathcal{X}$ always has the (StrongEpi,Mono) factorisation system [6, 1.61] and it has the (Epi,RegMono) factorisation system iff regular monos are closed under composition [5, 14.22]. In the latter case, we have RegMono=StrongMono [5, 14.14]. The proof of the theorem will be the same in both cases as it only uses the following properties shared by both factorisation systems: If $X_j \to X$ is a filtered colimit and all arrows of a cone $X_j \to Y$ are in $\mathcal{M}$ then the induced arrow $X \to Y$ is in $\mathcal{M}$, see [6, 1.60]. If $f \cdot g \in \mathcal{M}$, then $g \in \mathcal{M}$, see [5, 14.11].

Proof. There is a colimit $A_i \to A$ with $A_i \in \mathcal{A}_0$ and hence a colimit $PTSA_i \to LA$. To show $TSA \to SLA$ in $\mathcal{M}$, it suffices to show that $TSA \to SLA \to \prod_i SPTSA_i$ is in $\mathcal{M}$. Since η is natural and pointwise in $\mathcal{M}$ and since $\mathcal{M}$ is closed under products [5, 14.15], it suffices to show that $TSA \to \prod_i TSA_i$ is in $\mathcal{M}$. Observe that we have a filtered colimit $X_j \to SA$ and factoring $X_j \to SA$ as $X_j \to \bar{X}_j \to SA$ gives a filtered colimit $\bar{X}_j \to SA$ of arrows in $\mathcal{M}$ and hence also a filtered colimit $T\bar{X}_j \to TSA$ arrows in $\mathcal{M}$; due to [6, 1.62] it suffices to show that $T\bar{X}_j \to \prod_i TSA_i$ in $\mathcal{M}$. But this follows from (*).