Goldblatt-Thomason Theorems for Modal Intuitionistic Logics

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Abstract

We prove a Goldblatt-Thomason theorem for dialgebraic intuitionistic logics, and instantiate it to Goldblatt-Thomason theorems for a wide variety of modal intuitionistic logics from the literature.

Keywords: Modal logic, intuitionistic logic, Goldblatt-Thomason theorem.

1 Introduction

A prominent question in the study of (modal) logics and their semantics is what classes of frames can be defined as the class of frames satisfying some set of formulae. Such a class is usually called *axiomatic* or *modally definable*. A milestone result partially answering this question in the realm of classical normal modal logic is from Goldblatt and Thomason and dates back to 1974 [16]. It states that an elementary class of Kripke frames is axiomatic if and only if it reflects ultrafilter extensions and is closed under p-morphic images, generated subframes and disjoint unions. The proof in [16] relies on Birkhoff's variety theorem [4] and makes use of the algebraic semantics of the logic. A model-theoretic proof was provided almost twenty years later by Van Benthem [1].

A similar result for (non-modal) intuitionistic logic was proven by Rodenburg [30] (see also [15]), where the interpreting structures are *intuitionistic* Kripke frames and models. This, of course, requires analogues of the notions of p-morphic images, generated subframes, disjoint unions and ultrafilter extensions. While the first three carry over straightforwardly from the setting of classical normal modal logic, ultrafilters need to be replaced by *prime filters*.

In recent years, Goldblatt-Thomason style theorems (which we will simply refer to as "Goldblatt-Thomason theorems") for many other logics have been proven, including for positive normal modal logic [8], graded modal logic [31], modal extensions of Łukasiewicz finitely-valued logics [35], LE-logics [10], and modal logics with a universal modality [32]. A general Goldblatt-Thomason theorem for coalgebraic logics for Set-coalgebras was given in [22].

In the present paper we prove Goldblatt-Thomason theorems for modal intuitionistic logics. These include the extensions of intuitionistic logic with a normal modality [36,37,38], a monotone one [14, Sec. 6], a neighbourhood modality [11], and a strict implication modality [25,26,12]. For each we obtain:

A class $\mathscr{K}$ of frames closed under prime filter extensions is axiomatic if and only if it reflects prime filter extensions and is closed under disjoint unions, regular subframes and p-morphic images.

Instead of proving each of these results individually, we prove a more general Goldblatt-Thomason theorem for *dialgebraic intuitionistic logics*, merging techniques from [15] and [22]. We then apply this to specific instances.

Dialgebraic logic slightly generalises coalgebraic logic and was recently introduced in [18]. It provides a framework where modal logics are developed parametric in the signature of the language and a functor $\mathcal{T} : \mathsf{C}' \to \mathsf{C}$, where C' is some subcategory of C. While coalgebraic logics are too restrictive to describe modal intuitionistic logics (see e.g. [24, Rem. 8], [18, Sec. 2]), the additional flexibility of dialgebraic logic does allow us to model a number of them.

The paper is structured as follows. In Sec. 2 we recall a semantics for the extension of intuitionistic logic with a normal modality $\Box$ from [38]. Using this as running example, in Sec. 3 we recall the basics of dialgebraic logic and prove the Goldblatt-Thomason theorem. In particular, this yields a new Goldblatt-Thomason theorem for the logic and semantics from Sec. 2. In Sec. 4 we instantiate the general theorem to several more modal intuitionistic logics from the literature to obtain new Goldblatt-Thomason theorems.

2 Normal Modal Intuitionistic Logic

For future reference, we recall the extension of intuitionistic logic with a unary meet-preserving modality from Wolter and Zakharyaschev [37,38].

Definition 2.1 Denote the language of intuitionistic logic by $\mathbf{L}$, with proposition letters from some countably infinite set Prop. That is, $\mathbf{L}$ is generated by the grammar

$$\varphi ::= \top \mid \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi,$$

where $p \in \text{Prop}$. Write $\mathbf{L}_\Box$ for its extension with a unary operator $\Box$. Further, let $\boldsymbol{\mathcal{L}}$ be the intuitionistic propositional calculus, and let $\boldsymbol{\mathcal{L}}_\Box$ be the logic that arises from extending an axiomatisation for $\boldsymbol{\mathcal{L}}$ (that we assume includes uniform substitution) with the axioms and rule

$$\Box\top \leftrightarrow \top, \qquad \Box p \wedge \Box q \leftrightarrow \Box(p \wedge q), \qquad (p \leftrightarrow q)/(\Box p \leftrightarrow \Box q) \tag{1}$$

We write Pos for the category of posets and order-preserving functions. In this paper, we define an *intuitionistic Kripke frame* as a poset and we write Krip for the full subcategory of Pos whose morphisms are p-morphisms [2, Sec. 2.1.1]. (Sometimes intuitionistic Kripke frames are defined to be preorders. For the results presented in this paper there is no discernible difference.)

Definition 2.2 A $\Box$-*frame* is a triple $(X, \leq, R)$ where $(X, \leq)$ is an intuitionistic Kripke frame and R is a relation on X satisfying $(\leq \circ R \circ \leq) = R$.

Adding a valuation $V : \text{Prop} \to \mathcal{U}p(X, \leq)$ $(= \{a \subseteq X \mid x \in a \text{ and } x \leq y \text{ implies } y \in a\})$ yields a $\Box$-*model*, in which we can interpret $\mathbf{L}_\Box$-formulae. Proposition letters are interpreted via the valuation, intuitionistic connectives are interpreted as usual in the underlying intuitionistic Kripke frame and a state x satisfies $\Box\varphi$ if all its R-successors satisfy φ.

While morphisms are not defined in [37,38], there is an obvious choice:

Definition 2.3 A $\Box$-*morphism* form $(X, \leq, R)$ to $(X', \leq', R')$ is a function $f : X \to X'$ such that for $E \in \{\leq, R\}$ and for all $x, y \in X$ and $z' \in X'$:

- If xEy then $f(x)E'f(y)$;
- If $f(x)E'z'$ then $\exists z \in X$ such that xEz and $f(z) = z'$.

We write $\mathsf{WZ}\Box$ for the category of $\Box$-frames and -morphisms.

The algebraic semantics of $\mathcal{L}_\Box$ is given as follows.

Definition 2.4 A *Heyting algebra with operators* (HAO) is a pair $(A, \Box)$ of a Heyting algebra A and a function $\Box : A \to A$ satisfying $\Box\top = \top$ and $\Box a \wedge \Box b = \Box(a \wedge b)$ for all $a, b \in A$. Together with $\Box$-preserving Heyting homomorphisms, these constitute the category HAO.

We briefly recall some categories, functors and natural transformations.

Definition 2.5 DL and HA denote the categories of distributive lattices and Heyting algebras. Let up be the contravariant functor $\mathsf{Pos} \to \mathsf{DL}$ that sends a poset to the distributive lattice of its upsets and an order-preserving function f to f^{-1}. Write $pf : \mathsf{DL} \to \mathsf{Pos}$ for the contravariant functor sending $A \in \mathsf{DL}$ to the set of prime filters of A ordered by inclusion, and a homomorphism to its inverse image. These restrict to $up' : \mathsf{Krip} \to \mathsf{HA}$ and $pf' : \mathsf{HA} \to \mathsf{Krip}$.

Let $\eta : id_{\mathsf{Pos}} \to pf \circ up$ and $\theta : id_{\mathsf{DL}} \to up \circ pf$ be the natural transformations defined by $\eta_{(X,\leq)}(x) = \{a \in up(X, \leq) \mid x \in a\}$ and $\theta_A(a) = \{\mathfrak{p} \in pf A \mid a \in \mathfrak{p}\}$. (These are the units of the dual adjunction between Pos and DL.) Furthermore, θ restricts to the natural transformation $\theta' : id_{\mathsf{HA}} \to up' \circ pf'$.

Every $\Box$-frame $(X, \leq, R)$ yields a HAO $(up'(X, \leq), \Box_R)$ (called its *complex algebra*), with $\Box_R(a) = \{x \in X \mid xRy \text{ implies } y \in a\}$. Conversely, every HAO $(A, \Box)$ gives rise to a $\Box$-frame $(pf'A, \subseteq, R_\Box)$, where $\mathfrak{p}R_\Box\mathfrak{q}$ iff for all $a \in A$, $\Box a \in \mathfrak{p}$ implies $a \in \mathfrak{q}$. Concatenating these constructions yields:

Definition 2.6 The *prime filter extension* of a $\Box$-frame $(X, \leq, R)$ is the frame $(X^{pe}, \subseteq, R^{pe})$, where X^{pe} is the set of prime filters on $(X, \leq)$ and R^{pe} is defined by $\mathfrak{p}R^{pe}\mathfrak{q}$ iff for all $a \in up'(X, \leq)$, $\Box_R(a) \in \mathfrak{p}$ implies $a \in \mathfrak{q}$.

3 A General Goldblatt-Thomason Theorem

We restrict the framework of dialgebraic logic [18] to an intuitionistic base. Within this, we prove a Goldblatt-Thomason theorem. Throughout this section, we show how general constructions specialise to the normal modal intuitionistic logic from Sec. 2. Our focus on an intuitionistic propositional base allows us to augment the framework of dialgebraic logic from [18] in the following ways:

- In [18] a logic is identified via an initial object in some category, which plays the role of the Lindenbaum-Tarski algebra. Here we define logics explicitly, by means of an axiomatisation.
- Whereas proposition letters in [18] are regarded as predicate liftings, here we elevate them to a special status. This has two reasons: first, it simplifies the connection to (frames and models for) modal intuitionistic logics from the literature; second, they facilitate the use of Birkhoff's variety theorem.
- We give dialgebraic definitions of subframes, p-morphic images and disjoint unions, and corresponding preservation results.
- We give prime filter extensions for models (not just for frames).

We work towards a Goldblatt-Thomason theorem as follows. First we recall the use of dialgebras as frames for modal extensions of intuitionistic logic (Sec. 3.1), and we prove some invariance properties (Sec. 3.2). Then we describe algebraic semantics and prime filter extensions dialgebraically (Sec. 3.3 and 3.4). This culminates in the Goldblatt-Thomason theorem in Sec. 3.5.

3.1 Languages and Frames

Dialgebras were introduced by Hagino in [19] to describe data types. Here we use them to describe frames for modal intuitionistic logics.

Definition 3.1 Let $\mathcal{F}, \mathcal{G} : \mathsf{C} \to \mathsf{D}$ be functors. An $(\mathcal{F}, \mathcal{G})$-*dialgebra* is a pair (X, γ) where $X \in \mathsf{C}$ and $\gamma : \mathcal{F}X \to \mathcal{G}X$ is a D-morphism. An $(\mathcal{F}, \mathcal{G})$-*dialgebra morphism* from (X, γ) to (X', γ') is a C-morphism $f : X \to X'$ such that $\mathcal{G}f \circ \gamma = \gamma' \circ \mathcal{F}f$. They constitute the category $\mathsf{Dialg}(\mathcal{F}, \mathcal{G})$. In diagrams:

$$\text{objects:}\quad \begin{array}{c} \mathcal{F}X \\ \big\downarrow{\scriptstyle\gamma} \\ \mathcal{G}X \end{array} \qquad \text{arrows:}\quad \begin{array}{ccc} \mathcal{F}X & \xrightarrow{\mathcal{F}f} & \mathcal{F}X' \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\gamma'} \\ \mathcal{G}X & \xrightarrow{\mathcal{G}f} & \mathcal{G}X' \end{array}$$

We will be concerned with two classes of dialgebras. First, $(i, \mathcal{T})$-dialgebras, where $i : \mathsf{Krip} \to \mathsf{Pos}$ is the inclusion functor and $\mathcal{T} : \mathsf{Krip} \to \mathsf{Pos}$ is any functor, serve as frame semantics for our dialgebraic intuitionistic logics. Second, dialgebras for functors $\mathsf{HA} \to \mathsf{DL}$ will be used as algebraic semantics.

Example 3.2 Let $\mathcal{P}_{up} : \mathsf{Krip} \to \mathsf{Pos}$ be the functor that sends an intuitionistic Kripke frame $(X, \leq)$ to its set of upsets ordered by reverse inclusion, and a p-morphism $f : (X, \leq) \to (X', \leq')$ to $\mathcal{P}_{up}f : \mathcal{P}_{up}(X, \leq) \to \mathcal{P}_{up}(X', \leq') : a \mapsto f[a]$. Then identifying a relation R on X with the map $\gamma_R : (X, \leq) \to \mathcal{P}_{up}(X, \leq) : x \mapsto \{y \in X \mid xRy\}$ yields an isomorphism $\mathsf{WZ}\Box \cong \mathsf{Dialg}(i, \mathcal{P}_{up})$ [18, Sec. 2].

Modalities for $\mathsf{Dialg}(i, \mathcal{T})$ are defined via predicate liftings [18, Def. 5.7].

Definition 3.3 An *n-ary predicate lifting* for a functor $\mathcal{T} : \mathsf{Krip} \to \mathsf{Pos}$ is a natural transformation

$$\lambda : (\mathcal{U}p \circ i)^n \to \mathcal{U}p \circ \mathcal{T}.$$

Here $\mathcal{U}p : \mathsf{Pos} \to \mathsf{Set}$ is the contravariant functor that sends a poset to its set of upsets, and $(\mathcal{U}p \circ i)^n(X, \leq)$ is the n-fold product of $\mathcal{U}p(i(X, \leq))$ in Set.

Definition 3.4 Let Prop be a countably infinite set of proposition letters. For a set Λ of predicate liftings, define the language $\mathbf{L}(\Lambda)$ by the grammar

$$\varphi ::= \top \mid \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \heartsuit^\lambda(\varphi_1, \ldots, \varphi_n),$$

where p ranges over Prop and $\lambda \in \Lambda$ is n-ary.

Definition 3.5 Let Λ be a set of predicate liftings for $\mathcal{T} : \mathsf{Krip} \to \mathsf{Pos}$. An $(i, \mathcal{T})$-*model* $\mathfrak{M}$ is an $(i, \mathcal{T})$-dialgebra $\mathfrak{X} = (X, \leq, \gamma)$ with a valuation $V : \mathrm{Prop} \to \mathcal{U}p(X, \leq)$. Truth of $\varphi \in \mathbf{L}(\Lambda)$ at $x \in X$ is defined by

$$\mathfrak{M}, x \Vdash \top \quad \text{always}, \qquad \mathfrak{M}, x \Vdash \bot \quad \text{never}, \qquad \mathfrak{M}, x \Vdash p \quad \text{iff} \quad x \in V(p)$$
$$\begin{aligned}
\mathfrak{M}, x \Vdash \varphi \wedge \psi &\quad \text{iff} \quad \mathfrak{M}, x \Vdash \varphi \text{ and } \mathfrak{M}, x \Vdash \psi \\
\mathfrak{M}, x \Vdash \varphi \vee \psi &\quad \text{iff} \quad \mathfrak{M}, x \Vdash \varphi \text{ or } \mathfrak{M}, x \Vdash \psi \\
\mathfrak{M}, x \Vdash \varphi \to \psi &\quad \text{iff} \quad x \leq y \text{ and } \mathfrak{M}, y \Vdash \varphi \text{ imply } \mathfrak{M}, y \Vdash \psi \\
\mathfrak{M}, x \Vdash \heartsuit^\lambda(\varphi_1, \ldots, \varphi_n) &\quad \text{iff} \quad \gamma(x) \in \lambda_{(X, \leq)}(\llbracket \varphi_1 \rrbracket^{\mathfrak{M}}, \ldots, \llbracket \varphi_n \rrbracket^{\mathfrak{M}})
\end{aligned}$$

Here $\llbracket \varphi \rrbracket^{\mathfrak{M}} = \{x \in X \mid \mathfrak{M}, x \Vdash \varphi\}$. We write $\mathfrak{M} \Vdash \varphi$ if $\mathfrak{M}, x \Vdash \varphi$ for all $x \in X$ and $\mathfrak{X} \Vdash \varphi$ if $(\mathfrak{X}, V) \Vdash \varphi$ for all valuations V for $\mathfrak{X}$. If $\Phi \subseteq \mathbf{L}(\Lambda)$ then we say that Φ is *valid* on $\mathfrak{X}$, and write $\mathfrak{X} \Vdash \Phi$, if $\mathfrak{X} \Vdash \varphi$ for all $\varphi \in \Phi$. Also, let

$$\mathrm{Fr}\, \Phi = \{\mathfrak{X} \in \mathsf{Dialg}(i, \mathcal{T}) \mid \mathfrak{X} \Vdash \Phi\}.$$

We call a class $\mathscr{K} \subseteq \mathsf{Dialg}(i, \mathcal{T})$ *axiomatic* if $\mathscr{K} = \mathrm{Fr}\, \Phi$ for some $\Phi \subseteq \mathbf{L}(\Lambda)$.

Example 3.6 Since $\Box$-frames correspond to $(i, \mathcal{P}_{up})$-dialgebras, it is easy to see that $\Box$-models correspond to $(i, \mathcal{P}_{up})$-models. The modal operator $\Box$ can be induced by the predicate lifting $\lambda^\Box : \mathcal{U}p \circ i \to \mathcal{U}p \circ \mathcal{P}_{up}$ given by

$$\lambda^\Box_{(X, \leq)} : \mathcal{U}p(i(X, \leq)) \to \mathcal{U}p(\mathcal{P}_{up}(X, \leq)) : a \mapsto \{b \in \mathcal{P}_{up}(X, \leq) \mid b \subseteq a\}.$$

Indeed, if $\mathfrak{M} = (X, \leq, R, V)$ is a $\Box$-model and $(X, \leq, \gamma_R, V)$ the corresponding $(i, \mathcal{P}_{up})$-model then we have $x \Vdash \Box\varphi$ iff every R-successor of x satisfies φ, i.e. iff $\gamma_R(x) \subseteq \llbracket \varphi \rrbracket^{\mathfrak{M}}$. By definition the latter is equivalent to $\gamma_R(x) \in \lambda^\Box_{(X, \leq)}(\llbracket \varphi \rrbracket^{\mathfrak{M}})$.

Finally, we define morphisms between $(i, \mathcal{T})$-models.

Definition 3.7 An $(i, \mathcal{T})$-*model morphism* from $\mathfrak{M} = (\mathfrak{X}, V)$ to $\mathfrak{M}' = (\mathfrak{X}', V')$ is an $(i, \mathcal{T})$-dialgebra morphism $f : \mathfrak{X} \to \mathfrak{X}'$ such that $V = f^{-1} \circ V'$.

Proposition 3.8 *If $f : \mathfrak{M} \to \mathfrak{M}'$ is an $(i, \mathcal{T})$-model morphism, then for all states x of $\mathfrak{M}$ and $\varphi \in \mathbf{L}(\Lambda)$, we have $\mathfrak{M}, x \Vdash \varphi$ iff $\mathfrak{M}', f(x) \Vdash \varphi$.*

Proof. Let $\mathfrak{M} = (X, \leq, \gamma, V)$ and $\mathfrak{M}' = (X', \leq', \gamma', V')$. The proof proceeds by induction on the structure of φ. If $\varphi \in \mathrm{Prop}$ then the claim follows from the definition of an $(i, \mathcal{T})$-model morphism. The inductive cases for propositional connectives are routine, so we focus on the modal case. We restrict our attention to unary modalities, higher arities being similar. Compute:

$$\begin{aligned}
&\mathfrak{M}, x \Vdash \heartsuit^{\lambda}\varphi \\
&\qquad \text{iff} \quad \gamma(x) \in \lambda_{(X,\leq)}([\![\varphi]\!]^{\mathfrak{M}}) && \text{(Def. 3.5)} \\
&\qquad \text{iff} \quad \gamma(x) \in \lambda_{(X,\leq)}(f^{-1}([\![\varphi]\!]^{\mathfrak{M}'})) && \text{(Induction hypothesis)} \\
&\qquad \text{iff} \quad \gamma(x) \in \lambda_{(X,\leq)}((if)^{-1}([\![\varphi]\!]^{\mathfrak{M}'})) && \text{(Because } if = f\text{)} \\
&\qquad \text{iff} \quad \gamma(x) \in (\mathcal{T}f)^{-1}(\lambda_{(X',\leq')}([\![\varphi]\!]^{\mathfrak{M}'})) && \text{(Naturality of } \lambda\text{)} \\
&\qquad \text{iff} \quad (\mathcal{T}f)(\gamma(x)) \in \lambda_{(X',\leq')}([\![\varphi]\!]^{\mathfrak{M}'}) \\
&\qquad \text{iff} \quad \gamma'((if)(x)) \in \lambda_{(X',\leq')}([\![\varphi]\!]^{\mathfrak{M}'}) && (f \text{ is a dialgebra morphism}) \\
&\qquad \text{iff} \quad \mathfrak{M}', f(x) \Vdash \heartsuit^{\lambda}\varphi && \text{(Def. 3.5 and } if = f\text{)}
\end{aligned}$$

This proves the proposition. □

3.2 Disjoint Unions, Generated Subframes and p-Morphic Images

The category theoretic analogue of a disjoint union is a coproduct. For any $\mathcal{T} : \mathsf{Krip} \to \mathsf{Pos}$ the category $\mathsf{Dialg}(i, \mathcal{T})$ has coproducts because Krip has coproducts and i preserves them [6, Thm. 3.2.1]. So we define:

Definition 3.9 The *disjoint union* of a K-indexed family of $(i, \mathcal{T})$-dialgebras $\mathfrak{X}_k = (X_k, \leq_k, \gamma_k)$ is the coproduct $\coprod_{k \in K} \mathfrak{X}_k$ in $\mathsf{Dialg}(i, \mathcal{T})$.

Example 3.10 Let $(X_k, \leq_k, R_k)$ be a K-indexed set of $\Box$-frames, and $(X_k, \leq_k, \gamma_k)$ the corresponding $(i, \mathcal{P}_{up})$-dialgebras. The coproduct $\coprod_{k \in K}(X_k, \leq_k, \gamma_k)$ is given by $(X, \leq, \gamma)$, where $(X, \leq)$ is the coproduct of the intuitionistic Kripke frames $(X_k, \leq_k)$ (which is computed as in Set), and $\gamma : (X, \leq) \to \mathcal{P}_{up}(X, \leq)$ is given by $\gamma(x_k) = \gamma_k(x_k)$ (for $x_k \in X_k$). Transforming this back into a $\Box$-frame, we obtain $(X, \leq, R)$, with xRy iff there is a $k \in K$ with $x, y \in X_k$ and xR_ky. So this corresponds to the expected notion of disjoint union of $\Box$-frames.

Proposition 3.11 *Let $\mathfrak{X}_k = (X_k, \leq_k, \gamma_k)$ be a family of $(i, \mathcal{T})$-dialgebras indexed by some set K. Suppose $\mathfrak{X}_k \Vdash \varphi$ for all $k \in K$. Then $\coprod \mathfrak{X}_k \Vdash \varphi$.*

Proof. Let V be a valuation for $\coprod \mathfrak{X}_k$. Define the valuation V_k for $\mathfrak{X}_k$ by $V_k(p) = V(p) \cap X_k$. Then the coproduct inclusion maps $\kappa_k : (\mathfrak{X}_k, V_k) \to (\coprod \mathfrak{X}_k, V)$ are $(i, \mathcal{T})$-model morphisms, hence the assumption $\mathfrak{X}_k \Vdash \varphi$ for all $k \in K$ implies that $(\coprod \mathfrak{X}_k, V) \Vdash \varphi$. Since V was arbitrary, $\coprod \mathfrak{X}_k \Vdash \varphi$. □

Definition 3.12 Let $\mathfrak{X}' = (X', \leq', \gamma')$ and $\mathfrak{X} = (X, \leq, \gamma)$ be $(i, \mathcal{T})$-dialgebras.

(i) $\mathfrak{X}'$ is called a *generated subframe* of $\mathfrak{X}$ if there exists a p-morphism $f : \mathfrak{X}' \to \mathfrak{X}$ such that $f : (X', \leq') \to (X, \leq)$ is an embedding.

(ii) $\mathfrak{X}'$ is a *p-morphic image* of $\mathfrak{X}$ if there exists a surjective dialgebra morphism $\mathfrak{X} \to \mathfrak{X}'$.

Example 3.13 Guided by [5, Def. 2.5 and 3.13], we could define a generated sub-$\Box$-frame of a $\Box$-frame $(X, \leq, R)$ as a $\Box$-frame $(X', \leq', R')$ such that:

- $X' \subseteq X$ and $\leq' = (\leq \cap (X' \times X'))$ and $R' = (R \cap (X' \times X'))$;
- if $x \in X'$ and $x \leq y$ or xRy, then $y \in X'$.

With this definition, it can be shown that a $\Box$-frame $\mathfrak{X}'$ is isomorphic to a generated sub-$\Box$-frame of a $\Box$-frame $\mathfrak{X}$ if and only if the dialgebraic rendering of $\mathfrak{X}'$ is a generated subframe of the dialgebraic rendering of $\mathfrak{X}$ (as per Def. 3.12).

Proposition 3.14 *Let $\mathfrak{X}$ be an $(i, \mathcal{T})$-dialgebra such that $\mathfrak{X} \Vdash \varphi$.*

(i) *If $\mathfrak{X}'$ is a generated subframe of $\mathfrak{X}$ then $\mathfrak{X}' \Vdash \varphi$.*

(ii) *If $\mathfrak{X}'$ is a p-morphic image of $\mathfrak{X}$ then $\mathfrak{X}' \Vdash \varphi$.*

Proof. We prove the first item, the second item being similar. If $\mathfrak{X}' = (X', \leq', \gamma')$ is a generated subframe of $\mathfrak{X} = (X, \leq, \gamma)$ then there exists a $(i, \mathcal{T})$-dialgebra morphism $f : \mathfrak{X}' \to \mathfrak{X}$ that is an embedding of the underlying posets. Let V' be any valuation for $\mathfrak{X}'$. Define a valuation $V^{\uparrow}$ for $\mathfrak{X}$ by $V^{\uparrow}(p) = \{x \in X \mid \exists y \in V'(p) \text{ s.t. } f(y) \leq x\}$. Then the fact that f is an embedding implies that $V' = f^{-1}V^{\uparrow}$, and therefore $f : (\mathfrak{X}', V') \to (\mathfrak{X}, V^{\uparrow})$ is a dialgebra model morphism. The assumption that $\mathfrak{X} \Vdash \varphi$ together with Prop. 3.8 implies that $(\mathfrak{X}', V') \Vdash \varphi$. Since V' is arbitrary we find $\mathfrak{X}' \Vdash \varphi$. $\Box$

3.3 Axioms and Algebraic Semantics

In order to get intuition for the dialgebraic perspective of algebraic semantics, we observe that the category HAO is isomorphic to a category of dialgebras. In this case, we consider dialgebras for functors $\mathsf{HA} \to \mathsf{DL}$. Again, one of the functors is simply the inclusion functor, which we denote by $j : \mathsf{HA} \to \mathsf{DL}$.

Example 3.15 Let $\mathcal{K} : \mathsf{HA} \to \mathsf{DL}$ be the functor that sends a Heyting algebra A to the free distributive lattice generated by $\{\boxdot a \mid a \in A\}$ modulo $\boxdot\top = \top$ and $\boxdot a \wedge \boxdot b = \boxdot(a \wedge b)$, where a and b range over A. The action of $\mathcal{K}$ on a Heyting homomorphism $h : A \to A'$ is defined on generators by $\mathcal{K}h(\boxdot a) = \boxdot h(a)$. Then $\mathsf{HAO} \cong \mathsf{Dialg}(\mathcal{K}, j)$ [18, Exm. 3.3].

We denote generators by dotted boxes to distinguish them from the modality $\Box$. Observe that the relations defining $\mathcal{K}$ correspond to the axioms we want a normal box to satisfy. We investigate how to generalise this to the setting of some arbitrary set Λ of predicate liftings for a functor $\mathcal{T} : \mathsf{Krip} \to \mathsf{Pos}$.

Definition 3.16 A *rank-1 formula* in $\mathbf{L}(\Lambda)$ is a formula φ such that

- φ does not contain intuitionistic implication;
- each proposition letter appears in the scope of precisely one modal operator.

A *rank-1 axiom* is a formula of the form $\varphi \leftrightarrow \psi$, where φ, ψ are rank-1 formulae. It is called *sound* if it is valid in all $(i, \mathcal{T})$-dialgebras.

Let Ax be a collection of sound rank-1 axioms. Define the logic $\boldsymbol{\mathcal{L}}(\Lambda, \mathrm{Ax})$ as the smallest set of $\mathbf{L}(\Lambda)$-formulae containing Ax and an axiomatisation for intuitionistic logic, which is closed under modus ponens, uniform substitution, and

$$\frac{\varphi_1 \leftrightarrow \psi_1 \quad \cdots \quad \varphi_n \leftrightarrow \psi_n}{\heartsuit^\lambda(\varphi_1, \ldots, \varphi_n) \leftrightarrow \heartsuit^\lambda(\psi_1, \ldots, \psi_n)} \qquad \text{(congruence rule).}$$

Example 3.15 generalises as follows [18, Sec. 5].

Definition 3.17 Let Λ be a set of predicate liftings for $\mathcal{T}$ and Ax a set of sound rank-1 axioms for $\mathbf{L}(\Lambda)$. For a Heyting algebra A, define $\mathcal{L}^{(\Lambda,\mathrm{Ax})}A$ to be the free distributive lattice generated by $\{\dot\heartsuit^\lambda(a_1, \ldots, a_n) \mid \lambda \in \Lambda, a_i \in A\}$ modulo the axioms in Ax, where each occurrence of $\heartsuit$ is replaced by the formal generator $\dot\heartsuit$, $\leftrightarrow$ is replaced by $=$, and the proposition letters range over the elements of A. (This is well defined since the axioms in Ax are rank-1 axioms, which result in equations constructed from elements of the form $\dot\heartsuit(a_1, \ldots, a_n)$ and distributive lattice connectives.)

If $h : A \to A'$ is a Heyting homomorphism, define $\mathcal{L}^{(\Lambda,\mathrm{Ax})}h : \mathcal{L}^{(\Lambda,\mathrm{Ax})}A \to \mathcal{L}^{(\Lambda,\mathrm{Ax})}A'$ on generators by $\mathcal{L}^{(\Lambda,\mathrm{Ax})}h(\dot\heartsuit^\lambda(a_1, \ldots, a_n)) = \dot\heartsuit^\lambda(h(a_1), \ldots, h(a_n))$. Then $\mathcal{L}^{(\Lambda,\mathrm{Ax})} : \mathsf{HA} \to \mathsf{DL}$ defines a functor.

Again, we use a symbol with a dot in it to denote formal generators, and separate them from symbols in the language.

Example 3.18 Let $\Lambda = \{\lambda^\Box\}$, where $\lambda^\Box$ is the predicate lifting from Exm. 3.6, and write $\Box$ instead of $\heartsuit^{\lambda^\Box}$. Let Ax consist of the two axioms (not the rule) from (1), and note that these are both rank-1 axioms. Then the logic $\boldsymbol{\mathcal{L}}(\Lambda, \mathrm{Ax})$ coincides with $\boldsymbol{\mathcal{L}}_\Box$, and the functor obtained from the procedure in Def. 3.17 is naturally isomorphic to $\mathcal{K}$ from Exm. 3.15. (The only difference is the symbol used to represent the formal generators.)

The following observation allows us to use the Birkhoff variety theorem when proving the Goldblatt-Thomason theorem below.

Lemma 3.19 *Let $\mathcal{L}$ be obtained from predicate liftings and axioms via Def. 3.17. Then the category* $\mathsf{Dialg}(\mathcal{L}, j)$ *is a variety of algebras.*

Proof. It is known that the category HA of Heyting algebras is a variety of algebras. We add to its signature an n-ary operation symbol for each n-ary predicate lifting in Λ, and to the set of equations defining HA the equations obtained from Ax by replacing $\leftrightarrow$ with equality and proposition letters with variables. $\Box$

We can evaluate $\mathbf{L}(\Lambda)$-formulae in a $(\mathcal{L}^{(\Lambda,\mathrm{Ax})}, j)$-dialgebra (A, α) with an assignment of the proposition letters to elements of A. Intuitionistic connectives are interpreted as in the Heyting algebra A, and the interpretation of $\heartsuit^\lambda(\varphi_1, \ldots, \varphi_n)$ is given by $\alpha(\dot\heartsuit^\lambda((\!|\varphi_1|\!), \ldots, (\!|\varphi_n|\!)))$, where $(\!|\varphi_i|\!)$ is the interpretation of φ_i. We say that φ is valid in (A, α), and write $(A, \alpha) \models \varphi$, if φ evaluates to $\top$ under every assignment of the proposition letters.

This evaluation is closely related to the interpretation of formulae in $(i, \mathcal{T})$-dialgebras: a formula φ is valid in some $(i, \mathcal{T})$-dialgebra if and only if it is valid in some related algebra, called the complex algebra.

Definition 3.20 Define $\rho : \mathcal{L}^{(\Lambda,\mathrm{Ax})} \circ up' \to up \circ \mathcal{T}$ on generators by

$$\rho_{(X,\leq)}(\heartsuit^{\lambda}(a_1,\ldots,a_n)) = \lambda_{(X,\leq)}(a_1,\ldots,a_n).$$

Then ρ is a well defined transformation because Ax is assumed to be sound, and it is natural because predicate liftings are natural transformations.

It gives rise to a functor $(\cdot)^+ : \mathsf{Dialg}(i,\mathcal{T}) \to \mathsf{Dialg}(\mathcal{L}^{(\Lambda,\mathrm{Ax})}, j)$, which sends an $(i,\mathcal{T})$-dialgebra $(X,\leq,\gamma)$ to its *complex algebra* $(up'(X,\leq),\gamma^+)$, given by

$$\begin{tikzcd} \mathcal{L}^{(\Lambda,\mathrm{Ax})}(up'(X,\leq)) \arrow[r, "\rho_{(X,\leq)}"] \arrow[rr, bend right, "\gamma^+"'] & up(\mathcal{T}(X,\leq)) \arrow[r, "up\gamma"] & up(i(X,\leq)) = j(up'(X,\leq)). \end{tikzcd}$$

The action of $(\cdot)^+$ on an $(i,\mathcal{T})$-dialgebra morphism f is given by $f^+ = up' f$.

Example 3.21 Let $(X,\leq,R)$ be a $\Box$-frame and $(X,\leq,\gamma)$ the corresponding $(i,\mathcal{P}_{up})$-dialgebra. The complex algebra of $(X,\leq,\gamma)$ is the $(\mathcal{K},j)$-dialgebra $(up'(X,\leq),\gamma^+)$, where γ^+ is given by $\gamma^+(\boxdot a) = \gamma^{-1}(\lambda^{\Box}(a)) = \{x \in X \mid \gamma(x) \subseteq a\}$. Translating this to a HAO, we see that this corresponds precisely to the complex algebra of $(X,\leq,R)$ in the sense of Sec. 2.

Proposition 3.22 *Let $\mathfrak{X}$ be an $(i,\mathcal{T})$-dialgebra and $\varphi \in \mathbf{L}(\Lambda)$. Then we have*

$$\mathfrak{X} \Vdash \varphi \quad \textit{iff} \quad \mathfrak{X}^+ \models \varphi.$$

Proof. This follows from a routine induction on the structure of φ, where the base case follows from the fact that valuations for $\mathfrak{X}$ correspond bijectively to assignments of the proposition letters to elements of $\mathfrak{X}^+$. $\Box$

3.4 Prime Filter Extensions

The proof of the Goldblatt-Thomason theorem relies on Birkhoff's variety theorem and the connection between frame semantics and algebraic semantics of a logic. As we have seen above, every $\Box$-frame gives rise to a complex algebra, or, more generally, every $(i,\mathcal{T})$-dialgebra gives rise to a $(\mathcal{L},j)$-dialgebra. To transfer the variety theorem from $(\mathcal{L},j)$-dialgebras back to $(i,\mathcal{T})$-dialgebras, we need a functor $(\cdot)_+ : \mathsf{Dialg}(\mathcal{L},j) \to \mathsf{Dialg}(i,\mathcal{T})$ such that for each $(i,\mathcal{T})$-dialgebra $\mathfrak{X}$,

$$(\mathfrak{X}^+)_+ \Vdash \varphi \quad \text{implies} \quad \mathfrak{X} \Vdash \varphi. \tag{$\star$}$$

Assumption 3.23 *Throughout this subsection, let $\mathcal{T} : \mathsf{Krip} \to \mathsf{Pos}$ be a functor, Λ a set of predicate liftings for $\mathcal{T}$, and a set* Ax *of sound rank-1 axioms from $\mathbf{L}(\Lambda)$. Abbreviate $\mathcal{L} := \mathcal{L}^{(\Lambda,\mathrm{Ax})}$ and $\rho := \rho^{(\Lambda,\mathrm{Ax})}$.*

A functor $(\cdot)_+ : \mathsf{Dialg}(\mathcal{L},j) \to \mathsf{Dialg}(i,\mathcal{T})$ arises from a natural transformation τ in the same way as ρ induced a functor from frames to complex algebras. To stress its dependence on the choice of τ, we denote it by $(\cdot)_\tau$ instead of $(\cdot)_+$.

Definition 3.24 Let $\tau : pf \circ \mathcal{L} \to \mathcal{T} \circ pf'$ be a natural transformation. Then we define the contravariant functor $(\cdot)_\tau : \mathsf{Dialg}(\mathcal{L},j) \to \mathsf{Dialg}(i,\mathcal{T})$ on objects by sending a $(\mathcal{L},j)$-dialgebra $\mathcal{H} = (H,\alpha)$ to the $(i,\mathcal{T})$-dialgebra $\mathcal{H}_\tau$ given by

$$i(pf'H) = pf(jH) \xrightarrow{pf\,\alpha} pf(\mathcal{L}H) \xrightarrow{\tau_H} \mathcal{T}(pf'H).$$

For a $(\mathcal{L}, j)$-dialgebra morphism $h : \mathcal{H} \to \mathcal{H}'$ we define $h_\tau = pf'h : \mathcal{H}'_\tau \to \mathcal{H}_\tau$. Naturality of τ ensures that this is well defined.

We call $(\mathfrak{X}^+)_\tau$ the τ-prime filter extension of an $(i, \mathcal{T})$-dialgebra $\mathfrak{X}$ if τ satisfies a sufficient condition that ensures that $(\star)$ holds (by Prop. 3.27). This condition relies on the following variation of the adjoint mate of ρ.

Definition 3.25 Let $\rho : \mathcal{L} \circ up' \to up \circ \mathcal{T}$. Then we write $\rho^\flat$ for the natural transformation defined as the composition

$$\mathcal{T} \circ pf' \xrightarrow{\eta_{\mathcal{T} \circ pf'}} pf \circ up \circ \mathcal{T} \circ pf' \xrightarrow{pf\rho_{pf'}} pf \circ \mathcal{L} \circ up' \circ pf' \xrightarrow{pf(\mathcal{L}\theta')} pf \circ \mathcal{L},$$

where η and θ are defined as in Def. 2.5.

Definition 3.26 Let τ be a natural transformation such that $\rho^\flat \circ \tau = id_{pf \circ \mathcal{L}}$.

(i) Define $\mathfrak{pe}_\tau := (\cdot)_\tau \circ (\cdot)^+ : \mathsf{Dialg}(i, \mathcal{T}) \to \mathsf{Dialg}(i, \mathcal{T})$. We call $\mathfrak{pe}_\tau\, \mathfrak{X}$ the *τ-prime filter extension* of $\mathfrak{X} \in \mathsf{Dialg}(i, \mathcal{T})$.

(ii) The *τ-prime filter extension* of a model $\mathfrak{M} = (\mathfrak{X}, V)$ is $\mathfrak{pe}_\tau\, \mathfrak{M} := (\mathfrak{pe}_\tau\, \mathfrak{X}, V^{pe})$, where $V^{pe}(p) = \{\mathfrak{q} \in \mathfrak{pe}_\tau\, \mathfrak{X} \mid V(p) \in \mathfrak{q}\}$ for all $p \in \text{Prop}$.

Observe that the prime filter extension of an $(i, \mathcal{T})$-dialgebra $\mathfrak{X} = (X, \leq, \gamma)$ is of the form $\mathfrak{pe}_\tau\, \mathfrak{X} = (X^{pe}, \subseteq, \gamma^{pe})$, where X^{pe} denotes the set of prime filters of upsets of $(X, \leq)$ and γ^{pe} is computed using both ρ and τ.

We now show that τ-prime filter extensions satisfy $(\star)$.

Proposition 3.27 *Let τ be a natural transformation such that $\rho^\flat \circ \tau = id_{pf \circ \mathcal{L}}$, $\mathfrak{X} = (X, \leq, \gamma)$ an $(i, \mathcal{T})$-dialgebra, $\mathfrak{M} = (\mathfrak{X}, V)$ a model based on $\mathfrak{X}$, $\varphi \in \mathbf{L}(\Lambda)$.*

(i) *For all prime filters $\mathfrak{q} \in X^{pe}$ we have $\mathfrak{pe}_\tau\, \mathfrak{M}, \mathfrak{q} \Vdash \varphi$ iff $[\![\varphi]\!]^{\mathfrak{M}} \in \mathfrak{q}$.*

(ii) *For all states $x \in X$ we have $\mathfrak{M}, x \Vdash \varphi$ iff $\mathfrak{pe}_\tau\, \mathfrak{M}, \eta_{(X, \leq)}(x) \Vdash \varphi$.*

(iii) *If $\mathfrak{pe}_\tau\, \mathfrak{X} \Vdash \varphi$ then $\mathfrak{X} \Vdash \varphi$.*

Proof. The proof of the proposition is given in the appendix. $\square$

Example 3.28 Returning to our example of $\Box$-frames, we wish to find a natural transformation $\tau^\Box$ such that $(\rho^\Box)^\flat \circ \tau^\Box = id_{pf \circ \mathcal{L}^\Box}$.

Before defining $\tau^\Box$, let us get an idea of what $(\rho^\Box)^\flat$ looks like. Let A be a Heyting algebra and $Q \in pf(\mathcal{L}^\Box A)$. Since Q is determined by elements of the form $\boxdot a$ it contains, where $a \in A$, we pay special attention to these elements. For $D \in \mathcal{P}_{up} \circ pf'A$ and $a \in A$ we have

$$\begin{aligned} \boxdot a \in (\rho^\Box_A)^\flat(D) \quad &\text{iff} \quad \rho_{pf'A}((pf(\mathcal{L}^\Box \theta'_A))(\boxdot a)) \in \eta_{\mathcal{T}(pf'A)}(D) \\ &\text{iff} \quad \rho_{pf'A}(\boxdot \theta'_A(a)) \in \eta_{\mathcal{T}(pf'A)}(D) \\ &\text{iff} \quad D \in \rho_{pf'A}(\boxdot \theta'_A(a)) \\ &\text{iff} \quad D \subseteq \theta'_A(a) \end{aligned}$$

Guided by this we define $\tau : pf \circ \mathcal{L}^\Box \to \mathcal{P}_{up} \circ pf'$ on components by

$$\tau^\Box_A : pf(\mathcal{L}^\Box A) \to \mathcal{P}_{up}(pf'A) : Q \mapsto \{\mathfrak{p} \in pf'A \mid \forall a \in A, \boxdot a \in Q \text{ implies } a \in \mathfrak{p}\}$$

With this definition we can prove the following lemma, the proof of which can be found in the appendix.

Lemma 3.29 *$\tau^{\Box}$ is a natural transformation such that $(\rho^{\Box})^{\flat} \circ \tau^{\Box} = id_{pf \circ \mathcal{L}^{\Box}}$.*

Now suppose $(A, \Box)$ is a HAO, and $\mathcal{A} = (A, \alpha)$ its corresponding $(\mathcal{L}^{\Box}, j)$-dialgebra (with α given by $\alpha(\boxdot a) = \Box a$). We have $\mathcal{A}_{\tau} = (pf' A, \subseteq, \gamma)$, where

$$\gamma(\mathfrak{q}) = \{\mathfrak{p} \in pf' A \mid \forall a \in A, \boxdot a \in \alpha^{-1}(\mathfrak{q}) \text{ implies } a \in \mathfrak{p}\}.$$

Note that $\boxdot a \in \alpha^{-1}(\mathfrak{q})$ iff $\Box a = \alpha(\boxdot a) \in \mathfrak{q}$. Therefore, translating γ to a relation R_{γ}, we obtain: $\mathfrak{q} R_{\gamma} \mathfrak{p}$ iff $\Box a \in \mathfrak{q}$ implies $a \in \mathfrak{p}$ for all $a \in A$.

It follows that the $(i, \mathcal{T})$-dialgebra corresponding to the prime filter extension of a $\Box$-frame $(X, \leq, R)$ (as in Sec. 2) coincides with the $\tau^{\Box}$-prime filter extension of the dialgebraic rendering of $\mathfrak{X}$. So, modulo dialgebraic translation, prime filter extensions and $\tau^{\Box}$-prime filter extensions of $\Box$-frames coincide.

3.5 The Goldblatt-Thomason Theorem

Finally, we put our theory to work and prove a Goldblatt-Thomason theorem for dialgebraic intuitionistic logics. We work with the same assumptions as in Assum. 3.23. Additionally, we assume that we have a natural transformation $\tau : pf \circ \mathcal{L} \to \mathcal{T} \circ pf'$ such that $\rho^{\flat} \circ \tau = id_{pf \circ \mathcal{L}}$. This allows us to use Def. 3.26.

Definition 3.30 If $\Phi \subseteq \mathbf{L}(\Lambda)$ and $\mathcal{A} \in \mathsf{Dialg}(\mathcal{L}, j)$ then we write $\mathcal{A} \models \Phi$ if $\mathcal{A} \models \varphi$ for all $\varphi \in \Phi$. Besides, we let $\mathrm{Alg}\,\Phi = \{\mathcal{A} \in \mathsf{Dialg}(\mathcal{L}, j) \mid \mathcal{A} \models \Phi\}$ be the collection of $(\mathcal{L}, j)$-dialgebras satisfying Φ. We say that a class $\mathscr{C} \subseteq \mathsf{Dialg}(\mathcal{L}, j)$ is *axiomatic* if $\mathscr{C} = \mathrm{Alg}\,\Phi$ for some collection Φ of $\mathbf{L}(\Lambda)$-formulae.

Lemma 3.31 *$\mathscr{C} \subseteq \mathsf{Dialg}(\mathcal{L}, j)$ is axiomatic iff it is a variety of algebras.*

Proof. If $\mathsf{A} = \{\mathcal{A} \in \mathsf{Dialg}(\mathcal{L}, j) \mid \mathcal{A} \models \Phi\}$, then it is precisely the variety of algebras satisfying $\varphi^x \leftrightarrow \top$, where $\varphi \in \Phi$ and φ^x is the formula we get from φ by replacing the proposition letters with variables from some set S of variables. Conversely, suppose A is a variety of algebras given by a set E of equations using variables in S. For each equation $\varphi = \psi$ in E, let $(\varphi \leftrightarrow \psi)^p$ be the formula we get from replacing the variables in $\varphi \leftrightarrow \psi$ with proposition letters. Then we have $\mathsf{A} = \mathrm{Alg}\{(\varphi \leftrightarrow \psi)^p \mid \varphi = \psi \in E\}$. $\Box$

For a class $\mathscr{K}$ of $(i, \mathcal{T})$-dialgebras, write $\mathscr{K}^+ = \{\mathfrak{X}^+ \mid \mathfrak{X} \in \mathscr{K}\}$ for the collection of corresponding complex algebras. Also, if $\mathscr{C}$ is a class of algebras, then we write $H\mathscr{C}$, $S\mathscr{C}$ and $P\mathscr{C}$ for its closure under **h**omomorphic images, **s**ubalgebras and **p**roducts, respectively.

Lemma 3.32 *A class $\mathscr{K} \subseteq \mathsf{Dialg}(i, \mathcal{T})$ is axiomatic if and only if*

$$\mathscr{K} = \{\mathfrak{X} \in \mathsf{Dialg}(i, \mathcal{T}) \mid \mathfrak{X}^+ \in HSP(\mathscr{K}^+)\}. \tag{2}$$

Proof. Suppose $\mathscr{K}$ is axiomatic, i.e. $\mathscr{K} = \mathrm{Fr}\,\Phi$. Then it follows from Prop. 3.22 and the fact that H, S and P preserve validity of formulae that (2) holds. Conversely, suppose (2) holds. Since $HSP(\mathscr{K}^+)$ is a variety, Birkhoff's variety theorem states that it is of the from $\mathrm{Alg}\,\Phi$. It follows that $\mathscr{K} = \mathrm{Fr}\,\Phi$. $\Box$

We now have all the ingredients to prove the Goldblatt-Thomason theorem.

Theorem 3.33 *Let $\mathscr{K} \subseteq \mathsf{Dialg}(i, \mathcal{T})$ be closed under τ-prime filter extensions. Then $\mathscr{K}$ is axiomatic if and only if $\mathscr{K}$ reflects τ-prime filter extensions and is closed under disjoint unions, generated subframes and p-morphic images.*

Proof. The implication from left to right follows from Sec. 3.2 and Prop. 3.27. For the converse, by Lem. 3.32 it suffices to prove that $\mathscr{K} = \{\mathfrak{X} \in \mathsf{Dialg}(i, \mathcal{T}) \mid \mathfrak{X}^+ \in HSP(\mathscr{K}^+)\}$. So let $\mathfrak{X} = (X, \gamma) \in \mathsf{Dialg}(i, \mathcal{T})$ and suppose $\mathfrak{X}^+ \in HSP(\mathscr{K}^+)$. Then there are $\mathfrak{Z}_i \in \mathscr{K}$ such that $\mathfrak{X}^+$ is the homomorphic image of a sub-dialgebra $\mathcal{A}$ of the product of the $\mathfrak{Z}_i^+$. In a diagram:

$$\mathfrak{X}^+ \xleftarrow{\text{surjective}} \mathcal{A} \xrightarrow{\text{injective}} \prod \mathfrak{Z}_i^+$$

Since $\prod \mathfrak{Z}_i^+ = (\coprod \mathfrak{Z}_i)^+$, dually this yields

$$(\mathfrak{X}^+)_\tau \xrightarrow{\text{gen. subframe}} \mathcal{A}_\tau \xleftarrow{\text{p-morphic image}} \left((\coprod \mathfrak{Z}_i)^+\right)_\tau$$

We have $\coprod \mathfrak{Z}_i \in \mathscr{K}$ because $\mathscr{K}$ is closed under coproducts, and $\left((\coprod \mathfrak{Z}_i)^+\right)_\tau \in \mathscr{K}$ because $\mathscr{K}$ is closed under prime filter extensions. Then $\mathcal{A}_\tau \in \mathscr{K}$ and $(\mathfrak{X}^+)_\tau \in \mathscr{K}$ because $\mathscr{K}$ is closed under p-morphic images and generated subframes. Finally, since $\mathscr{K}$ reflects prime filter extensions we find $\mathfrak{X} \in \mathscr{K}$. $\square$

Circling back to $\Box$-frames, it follows from Lem. 3.29 and Thm. 3.33 that:

Theorem 3.34 *Suppose $\mathscr{K} \subseteq \mathsf{WZ}\Box$ is closed under prime filter extensions. Then $\mathscr{K}$ is axiomatic if and only if it reflects prime filter extensions and is closed under disjoint unions, generated subframes and p-morphic images.*

4 Applications

In each of the following subsection we recall a modal intuitionistic logic and model it dialgebraically. We use this to derive a notion of prime filter extension and we apply Thm. 3.33 to obtain a Goldblatt-Thomason theorem.

4.1 Goldblatt's Geometric Modality I

The extension of intuitionistic logic with a monotone modality, here denoted by $\triangle$, was first studied by Goldblatt in [14, Sec. 6]. It is closely related to its classical counterpart [9,20,21], except that the underlying propositional logic is intuitionistic. A dialgebraic perspective was given in [18, Sec. 8].

Let $\mathbf{L}_\triangle$ denote the language of intuitionistic logic extended with a unary operator $\triangle$, and write $\boldsymbol{\mathcal{L}}_\triangle$ for the logic obtained from extending intuitionistic logic $\boldsymbol{\mathcal{L}}$ with the axiom $\triangle(p \wedge q) \to \triangle p$ and the congruence rule for $\triangle$.

Definition 4.1 An *intuitionistic monotone frame* (or *IM-frame*) is a triple $(X, \leq, N)$ where $(X, \leq)$ is an intuitionistic Kripke frame and N is a function that assigns to each $x \in X$ a collection of upsets of $(X, \leq)$ such that:

- if $a \in N(x)$ and $a \subseteq b \in \mathcal{U}p(X, \leq)$, then $b \in N(x)$;
- if $x \leq y$ then $N(x) \subseteq N(y)$.

An *intuitionistic monotone frame morphism* (IMF-morphism) from $(X_1, \leq_1, N_1)$ to $(X_2, \leq_2, N_2)$ is a p-morphism $f : (X_1, \leq_1) \to (X_2, \leq_2)$ such that

$f^{-1}(a_2) \in N_1(x_1)$ iff $a_2 \in N_2(f(x_1))$ for all $x_1 \in X_1$ and $a_2 \in \mathcal{U}p(X_2, \leq_2)$. We write Mon for the category of intuitionistic monotone frames and morphisms.

An *intuitionistic monotone model* is a tuple $\mathfrak{M} = (X, \leq, N, V)$ such that $(X, \leq, N)$ is an intuitionistic monotone frame and $V : \mathrm{Prop} \to \mathcal{U}p(X, \leq)$ is a valuation. The interpretation of $\mathbf{L}_\triangle$-formulae at a state x in $\mathfrak{M}$ is defined recursively, where the propositional cases are as usual and $\mathfrak{M}, x \Vdash \triangle\varphi$ iff $[\![\varphi]\!]^{\mathfrak{M}} \in N(x)$. We now take a dialgebraic perspective.

Definition 4.2 For an intuitionistic Kripke frame $(X, \leq)$, define

$$\mathcal{M}(X, \leq) = \{W \subseteq \mathcal{U}p(X, \leq) \mid \text{ if } a \in W \text{ and } a \subseteq b \in \mathcal{U}p(X, \leq) \text{ then } b \in W\}$$

ordered by inclusion. For a p-morphism $f : (X_1, \leq_1) \to (X_2, \leq_2)$, let

$$\mathcal{M}f : \mathcal{M}(X_1, \leq_1) \to \mathcal{M}(X_2, \leq_2) : W \mapsto \{a_2 \in \mathcal{U}p(X_2, \leq_2) \mid f^{-1}(a_2) \in W\}.$$

Then $\mathcal{M} : \mathsf{Krip} \to \mathsf{Pos}$ defines a functor.

Theorem 4.3 ([18], Thm. 8.3) *We have* $\mathsf{Mon} \cong \mathsf{Dialg}(i, \mathcal{M})$.

Translating the dialgebraic notion of disjoint union to IM-frames gives:

Definition 4.4 Let $\{(X_k, \leq_k, N_k) \mid k \in K\}$ be a K-indexed set of IM-frames. The disjoint union $\coprod_{k \in K}(X_k, \leq_k, N_k)$ is the frame $(X, \leq, N)$ where $(X, \leq)$ is the disjoint union of the intuitionistic Kripke frames $(X_k, \leq_k)$, and N is given by $a \in N(x_k)$ iff $a \cap X_k \in N_k(x_k)$ for all $a \in \mathcal{U}p(X, \leq)$ and $x_k \in X_k$.

Definition 4.5 An IM-frame $\mathfrak{X}'$ is a *generated subframe* of an IM-frame $\mathfrak{X}$ if there exists an IMF-morphism $\mathfrak{X}' \to \mathfrak{X}$ that is an embedding of posets, and $\mathfrak{X}'$ is a *p-morphic image* of $\mathfrak{X}$ if there is a surjective IMF-morphism $\mathfrak{X} \to \mathfrak{X}'$.

The modal operator $\triangle$ can be introduced by the predicate lifting $\lambda^\triangle : \mathcal{U}p \circ i \to \mathcal{U}p \circ \mathcal{M}$ given by

$$\lambda^\triangle_{(X,\leq)}(a) = \{W \in \mathcal{M}(X, \leq) \mid a \in W\}.$$

With $\mathrm{Ax} = \{\triangle(a \wedge b) \wedge \triangle a \leftrightarrow \triangle(a \wedge b)\}$ we have $\boldsymbol{\mathcal{L}}_\triangle = \boldsymbol{\mathcal{L}}(\{\lambda^\triangle\}, \mathrm{Ax})$. Its algebraic semantics is given by $(\mathcal{L}^\triangle, j)$-dialgebras, where $\mathcal{L}^\triangle : \mathsf{HA} \to \mathsf{DL}$ is the functor sending A to the free distributive lattice generated by $\{\blacktriangle a \mid a \in H\}$ modulo $\blacktriangle(a \wedge b) \leq \blacktriangle b$. The corresponding natural transformation $\rho^\triangle : \mathcal{L}^\triangle \circ up' \to up \circ \mathcal{M}$ is defined on generators by $\rho^\triangle_{(X,\leq)}(a) = \{W \in \mathcal{M}(X, \leq) \mid a \in W\}$.

Towards prime filter extensions and a Goldblatt-Thomason theorem we need to define a right inverse τ of $(\rho^\triangle)^\flat$. To garner inspiration we investigate what $(\rho^\triangle)^\flat_A : \mathcal{M}(pf'A) \to pf(\mathcal{L}^\triangle A)$ looks like for $A \in \mathsf{HA}$. We have

$$\blacktriangle a \in (\rho^\triangle)^\flat_A(W) \quad \text{iff} \quad \rho_{pf'A}(\blacktriangle\theta'_A(a)) \in \eta_{\mathcal{M} \circ pf'A}(W) \quad \text{iff} \quad \theta'_A(a) \in W$$

for all $W \in \mathcal{M}(pf'A)$ and $a \in A$. (Recall that $\theta'_A(a) = \{\mathfrak{q} \in pf'A \mid a \in \mathfrak{q}\}$.)

Definition 4.6 Let $A \in \mathsf{HA}$. We call $D \in up'(pf'A)$ *closed* if $D = \bigcap\{\theta'_A(a) \mid a \in A \text{ and } D \subseteq \theta'_A(a)\}$, and *open* if $D = \bigcup\{\theta'_A(a) \mid a \in A \text{ and } \theta'_A(a) \subseteq D\}$.

(Indeed, this coincides with closed and open upsets of $pf'A$, conceived of as an Esakia space [2, Sec. 2.3.3].) Upsets of the form $\theta'_A(a)$ are closed *and* open.

Definition 4.7 For a Heyting algebra A, define $\tau_A : pf \circ \mathcal{L}^{\triangle} A \to \mathcal{M} \circ pf' A$ as follows. Let $Q \in pf(\mathcal{L}^{\triangle} A)$ and $D \in \mathcal{U}p(pf'A)$, and define:

- If $D = \theta'_A(a)$ for some $a \in A$, then $\theta'_A(a) \in \tau_A(Q)$ if $\triangle a \in Q$;
- If D is closed then $D \in \tau_A(Q)$ if for all $a \in A$, $D \subseteq \theta'_A(a)$ implies $\triangle a \in Q$.
- For other D, $D \in \tau_A(Q)$ if there is a closed upset $C \subseteq D$ such that $C \in \tau_A(Q)$.

It is easy to see that $\tau^{\triangle}_A$ is an order-preserving function, i.e. a morphism in Pos. The next lemma states that $\tau^{\triangle}$ is a natural transformation. We postpone the unexciting proof to the appendix.

Lemma 4.8 *The transformation τ from Def. 4.7 is natural. Moreover, $(\rho^{\triangle})^{\flat}_A \circ \tau^{\triangle}_A = id(pf(\mathcal{L}^{\triangle} A))$ for every Heyting algebra A.*

Translating the dialgebraic definition of a prime filter extension to IM-frames gives a definition of prime filter extension for IM-frames. We emphasise that this definition relies on $\tau^{\triangle}$. In the next section we derive a different notion of prime filter extension for IM-frames, with its own Goldblatt-Thomason theorem.

Definition 4.9 The *$\tau^{\triangle}$-prime filter extension* of an IM-frame $(X, \leq, N)$ is the IM-frame $(X^{pe}, \subseteq, N^{pe})$, where N^{pe} is given as follows. Let $\triangle_N(a) = \{x \in X \mid a \in N(x)\}$, and for $\mathfrak{q} \in X^{pe}$ and $D \in \mathcal{U}p(X^{pe}, \subseteq)$ define:

- If $a \in up'(X, \leq)$, then $\theta'_A(a) \in N^{pe}(\mathfrak{q})$ if $\triangle_N a \in \mathfrak{q}$;
- If D is closed then $D \in N^{pe}(\mathfrak{q})$ if $\theta'_A(a) \in N^{pe}(\mathfrak{q})$ for all $\theta'_A(a)$ containing D;
- For any D, $D \in N^{pe}(\mathfrak{q})$ if there is a closed $C \subseteq D$ such that $C \in N^{pe}(\mathfrak{q})$.

Now Thm. 3.33 instantiates to:

Theorem 4.10 *Suppose $\mathscr{K}$ is a class of IM-frames closed under $\tau^{\triangle}$-prime filter extensions. Then $\mathscr{K}$ is axiomatic iff it reflects $\tau^{\triangle}$-prime filter extensions and is closed under disjoint unions, generated subframes and p-morphic images.*

4.2 Goldblatt's Geometric Modality II

We substantiate the claim that a logic may have several notions of prime filter extension by giving a different right-inverse of $(\rho^{\triangle})^{\flat}$ from Sec. 4.1. The setup is the same as in Sec. 4.1, so we proceed by defining a right-inverse of $(\rho^{\triangle})^{\flat}$.

Definition 4.11 For a Heyting algebra A, define $\sigma_A : pf \circ \mathcal{L}^{\triangle} A \to \mathcal{M} \circ pf' A$ by sending $Q \in pf(\mathcal{L}^{\triangle} A)$ to $\sigma_A(Q)$, where:

- For open upsets D, let $D \in \sigma_A(Q)$ if $\exists a \in A$ s.t. $\triangle a \in Q$ and $\theta'_A(a) \subseteq D$;
- For any other upset D, let $D \in \sigma_A(Q)$ if all open supersets of D are in $\sigma_A(Q)$.

Similar to Lem. 4.8 we can prove the following.

Lemma 4.12 *$\sigma = (\sigma_A)_{A \in \mathsf{HA}} : pf \circ \mathcal{L}^{\triangle} \to \mathcal{M} \circ pf'$ is a natural transformation, and for every Heyting algebra A, we have $\rho^{\flat}_A \circ \sigma_A = id_{pf(\mathcal{L}^{\triangle} A)}$.*

Now σ yields a different notion of prime filter extension, the precise definition of which we leave to the reader. Thm. 3.33 yields a Goldblatt-Thomason theorem with respect to this different notion of prime filter extension.

Theorem 4.13 *Let $\mathscr{K}$ be a class of IM-frames closed under σ-prime filter extensions. Then $\mathscr{K}$ is axiomatic iff it reflects σ-prime filter extensions and is closed under disjoint unions, generated subframes and p-morphic images.*

4.3 Non-Normal Intuitionistic Modal Logic

Neighbourhood semantics is used to accommodate for non-normal modal operators [33,27,9,28]. Dalmonte, Grellois and Olivett recently put forward an intuitionistic analogue [11] to interpret the extension of intuitionistic logic with unary modalities $\Box$ and $\Diamond$ which a priori do not satisfy any interaction axioms.

The ordered sets underlying the neighbourhood semantics from [11] are allowed to be preorders. Conforming to our general framework, we shall assume them to be posets. However, as mentioned in the introduction, we can obtain exactly the same (dialgebraic) results when replacing posets with preorders.

We use $\wp$ to denote the (covariant) powerset functors on Set.

Definition 4.14 A *coupled intuitionistic neighbourhood frame* or *CIN-frame* is a tuple $(X, \leq, N_\Box, N_\Diamond)$ such that $(X, \leq)$ is an intuitionistic Kripke frame and $N_\Box, N_\Diamond$ are functions $X \to \wp\wp X$ such that for all $x, y \in X$:

$$x \leq y \quad \text{implies} \quad N_\Box(x) \subseteq N_\Box(y) \quad \text{and} \quad N_\Diamond(x) \supseteq N_\Diamond(y).$$

A CIN-morphism $f : (X, \leq, N_\Box, N_\Diamond) \to (X', \leq', N'_\Box, N'_\Diamond)$ is a p-morphism $f : (X, \leq) \to (X', \leq')$ where for all $N \in \{N_\Box, N_\Diamond\}$, $x \in X$, $a' \in \wp X'$, $f^{-1}(a') \in N(x)$ iff $a' \in N'(f(x))$. CIN denotes the category of CIN-frames and -morphisms.

The language $\mathbf{L}_{\Box\Diamond}$ extending the intuitionistic language with unary modalities $\Box$ and $\Diamond$ can be interpreted in models based on CIN-frames, where

$$x \Vdash \Box\varphi \quad \text{iff} \quad [\![\varphi]\!] \in N_\Box(x), \qquad x \Vdash \Diamond\varphi \quad \text{iff} \quad X \setminus [\![\varphi]\!] \notin N_\Diamond(x).$$

We now view this dialgebraically:

Definition 4.15 Define $\mathcal{N} : \mathsf{Krip} \to \mathsf{Pos}$ on objects $(X, \leq)$ by $\mathcal{N}(X, \leq) = (\wp\wp X, \subseteq) \times (\wp\wp X, \supseteq)$, and on morphisms $f : (X, \leq) \to (X', \leq')$ by

$$\mathcal{N} f(W_1, W_2) = \big(\{a'_1 \in \wp X' \mid f^{-1}(a'_1) \in W_1\}, \{a'_2 \in \wp X' \mid f^{-1}(a'_2) \in W_2\}\big).$$

Theorem 4.16 *We have* $\mathsf{CIN} \cong \mathsf{Dialg}(i, \mathcal{N})$.

Proof. The isomorphism on objects is obvious. The isomorphism on morphisms follows from a computation similar to that in the proof of Thm. 4.3. $\Box$

The modal operators $\Box, \Diamond$ are induced by $\lambda^\Box, \lambda^\Diamond : \mathcal{U}p \circ i \to \mathcal{U}p \circ \mathcal{N}$, where

$$\lambda^\Box_{(X,\leq)}(a) = \{(W_1, W_2) \in \mathcal{N}(X, \leq) \mid a \in W_1\}$$
$$\lambda^\Diamond_{(X,\leq)}(a) = \{(W_1, W_2) \in \mathcal{N}(X, \leq) \mid X \setminus a \notin W_2\}$$

Unravelling the definition of a disjoint union of (the dialgebraic renderings of) CIN-frames shows that it is computed similar to Def. 4.4. Generated subframes and p-morphic images are defined by means of CIN-morphisms.

Since $\Box$ and $\Diamond$ only satisfy the congruence rule, the algebraic semantics is given by dialgebras for the functor $\mathcal{L}^{\Box\Diamond} : \mathsf{HA} \to \mathsf{DL}$ that sends A to the free distributive lattice generated by $\{\boxdot a, \diamonddot a \mid a \in A\}$. The induced natural transformation $\rho^{\Box\Diamond} : \mathcal{L}^{\Box\Diamond} \circ up' \to up \circ \mathcal{N}$ is defined on components via $\rho^{\Box\Diamond}_{(X,\leq)}(\boxdot a) = \lambda^{\Box}_{(X,\leq)}(a)$ and $\rho^{\Box\Diamond}_{(X,\leq)}(\diamonddot a) = \lambda^{\Diamond}_{(X,\leq)}(a)$. Akin to Sec. 4.1 we find $\boxdot a \in (\rho^{\Box\Diamond}_A)^\flat(W_1, W_2)$ iff $\theta'_A(a) \in W_1$ and $\diamonddot a \in (\rho^{\Box\Diamond}_A)^\flat(W_1, W_2)$ iff $pf'A \setminus \theta'_A(a) \notin W_1$ for all $A \in \mathsf{HA}$, $(W_1, W_2) \in \mathcal{N}(pf'A)$ and $a \in A$.

Definition 4.17 For a Heyting algebra A, define

$$\tau_A : pf(\mathcal{L}^{\Box\Diamond} A) \to \mathcal{N}(pf'A) : Q \mapsto \big(\{\theta'_A(a) \mid \boxdot a \in Q\}, \{pf'A \backslash \theta'_A(a) \mid \diamonddot a \notin Q\}\big).$$

Then $\tau = (\tau_A)_{A \in \mathsf{HA}}$ defines a natural transformation $pf \circ \mathcal{L}^{\Box\Diamond} \to \mathcal{N} \circ pf'$. It follows from the definitions that $(\rho^{\Box\Diamond})^\flat \circ \tau = id_{pf \circ \mathcal{L}^{\Box\Diamond}}$. We get the following definition of τ-prime filter extensions and Goldblatt-Thomason theorem.

Definition 4.18 The τ-prime filter extension of a CIN-frame $\mathfrak{X} = (X, \leq, N_\Box, N_\Diamond)$ is given by $\mathfrak{pe}_\tau\,\mathfrak{X} = (X^{pe}, \subseteq, N^{pe}_\Box, N^{pe}_\Diamond)$, where for $\mathfrak{q} \in X^{pe}$ we have

$$N^{pe}_\Box(\mathfrak{q}) = \{\theta'_{up'(X,\leq)}(a) \in \wp X^{pe} \mid a \in up(X,\leq) \text{ and } \Box_N(a) \in \mathfrak{q}\}$$
$$N^{pe}_\Diamond(\mathfrak{q}) = \{X^{pe} \setminus \theta'_{up'(X,\leq)}(a) \in \wp X^{pe} \mid a \in up(X,\leq) \text{ and } \Diamond_N(a) \in \mathfrak{q}\}$$

Here $\Box_N(a) = \{x \in X \mid a \in N_\Box(x)\}$ and $\Diamond_N(a) = \{x \in X \mid X \setminus a \notin N_\Diamond(x)\}$.

Theorem 4.19 *Let $\mathscr{K}$ be a class of CIN-frames closed under τ-prime filter extensions. Then $\mathscr{K}$ is axiomatic iff it reflects τ-prime filter extensions and is closed under disjoint unions, generated subframes and p-morphic images.*

4.4 Heyting-Lewis Logic

Finally we discuss Heyting-Lewis logic, the extension of intuitionistic logic with a binary strict implication operator $⥽$ [25,26,12].

Definition 4.20 A *strict implication frame* is a tuple $(X, \leq, R_s)$, where $(X, \leq)$ is an intuitionistic Kripke frame and R_s is a relation on X such that $x \leq y R_s z$ implies $x R_s z$. Morphisms between them are functions that are p-morphisms with respect to both orders. Models are defined as expected, and $⥽$ is interpreted via

$$x \Vdash \varphi ⥽ \psi \quad \text{iff} \quad \text{for all } y \in X, \text{ if } x R_s y \text{ and } y \Vdash \varphi \text{ then } y \Vdash \psi.$$

Strict implication frames can be modelled as $(i, \mathcal{P}_s)$-dialgebras, where $\mathcal{P}_s : \mathsf{Krip} \to \mathsf{Pos}$ is the functor that sends $(X, \leq)$ to $(\wp X, \subseteq)$ ($\wp$ denotes the covariant powerset functor) and a p-morphism f to $\wp f$. The modality $⥽$ can then be defined via the binary predicate lifting $\lambda^{⥽}$, given on components by

$$\lambda^{⥽}_{(X,\leq)}(a, b) = \{c \in \mathcal{P}_s(X, \leq) \mid c \cap a \subseteq b\}.$$

Disjoint unions, generated subframes and p-morphic images are defined as for $\Box$-frames.

The algebraic semantics for this logic given in [12, Def. III.1] can be modelled dialgebraically in a similar way as we have seen above. Computation of the natural transformation $\rho^{\dashv}$ is, by now, routine. Examining the proof of the duality for Heyting-Lewis logic sketched in [12, Section III-D], we can compute a one-sided inverse τ to $(\rho^{\dashv})^\flat$. We suppress the details, but do give the resulting notion of prime filter extension:

Definition 4.21 The *prime filter extension* of a strict implication frame $(X, \leq, R_s)$ is given by the frame $(X^{pe}, \subseteq, R_s^{pe})$, with R_s^{pe} defined by

$$\mathfrak{p} R_s^{pe} \mathfrak{q} \quad \text{iff} \quad \forall a, b \in up(X, \leq), \text{if } a \dashv_R b \in \mathfrak{p} \text{ and } a \in \mathfrak{q} \text{ then } b \in \mathfrak{q}$$

where $a \dashv_R b = \{x \in X \mid R[x] \cap a \subseteq b\}$.

With this notion of prime filter extension, Thm. 3.33 instantiates to:

Theorem 4.22 *A class $\mathscr{K}$ of strict implication frames that is closed under prime filter extensions is axiomatic iff it reflects prime filter extensions and is closed under disjoint unions, generated subframes and p-morphic images.*

5 Conclusions

We have given a general way to obtain Goldblatt-Thomason theorems for modal intuitionistic logics, using the framework of dialgebraic logic. Subsequently, we applied the general result to several concrete modal intuitionistic logics. The results in this paper can be generalised in several directions.

More applications. The general Goldblatt-Thomason theorem can also be instantiated to $\Diamond$-frames and $\Box\Diamond$-frames [38]. Using preorders instead of posets, we can obtain Goldblatt-Thomason theorems for ((strictly) condensed) $H\Box$-frames and $H\Box\Diamond$ frames used by Božić and Došen [7].

More base logics. The framework of dialgebraic logic is not restricted to an intuitionistic base. Generalising the results from this paper, we can obtain a general Goldblatt-Thomason theorem that also covers modal bi-intuitionistic logics [17] and modal lattice logics [3]. Moreover, this would also cover coalgebraic logics over a classical and a positive propositional base. The results in this paper can be generalised to dialgebraic logics for different base logics. This would give rise to Goldblatt-Thomason

Other modal intuitionistic logics The results in the paper do not apply to the modal intuitionistic logics investigated by Fischer Servi [13], Plotkin and Sterling [29], and Simpson [34], because these formalisms are not covered by the dialgebraic approach. It would be interesting to see if similar techniques can be applied to these logics to still prove Goldblatt-Thomason theorems.

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Appendix

A Omitted proofs

We use the following lemma in the proof of Prop. 3.27.

Lemma A.1 *Let τ be a natural transformation such that $\rho^\flat \circ \tau = id_{pf \circ \mathcal{L}}$, and $\mathcal{A} = (A, \alpha) \in \mathsf{Dialg}(\mathcal{L}, j)$. Then $\theta'_A : A \to up'(pf'H)$ defines a $(\mathcal{L}, j)$-dialgebra morphism from $\mathcal{A}$ to $(\mathcal{A}_\tau)^+$.*

Proof. This is similar to [23, Theorem 6.4(1)]. We repeat the argument here.

Let $\mathcal{A} = (A, \alpha)$ be a $(\mathcal{L}, j)$-dialgebra. Then $(\mathcal{A}_\tau)^+$ is given by the composition

$$\mathcal{L}(up'(pf'A)) \xrightarrow{\rho_{pf'A}} up(\mathcal{T}(pf'A)) \xrightarrow{up\tau_A} up(pf(\mathcal{L}A)) \xrightarrow{up \circ pf\alpha} up(pf(jA)) = j(up'(pf'A))$$

In order to show that θ'_A is a morphism from $\mathcal{A}$ to $(\mathcal{A}_\tau)^+$ we need to show that the outer shell of the following diagram commutes:

$$\begin{tikzcd}
\mathcal{L}A \arrow[rrrr, "\alpha"] \arrow[d, "\mathcal{L}\theta'_A"'] \arrow[drr, "\theta_{\mathcal{L}A}"] & & & & jA \arrow[dl, "\theta_{jA}"'] \arrow[d, "j\theta'_A"] \\
\mathcal{L}(up'(pf'A)) \arrow[r, "\rho_{pf'A}"'] & up(\mathcal{T}(pf'A)) \arrow[r, "up\tau_A"'] & up(pf(\mathcal{L}A)) \arrow[r, "up(pf\alpha)"'] & up(pf(jA)) \arrow[r, equal] & j(up'(pf'A))
\end{tikzcd}$$

The right triangle commutes by definition. The middle square commutes by naturality of θ. So we are left to prove that $\theta_{\mathcal{L}A} = up\tau_A \circ \rho_{pf'A} \circ \mathcal{L}\theta'_A$.

Since $\rho^\flat \circ \tau = \mathrm{id}$, hence $up\tau \circ up\rho^\flat = \mathrm{id}_{up}$, it suffices to prove that $up\rho^\flat \circ \theta_{\mathcal{L}A} = \rho_{pf'A} \circ \mathcal{L}\theta'_A$. (The result then follows from composing both sides with $up\tau_A$ on the left.) This is precisely the outer shell of the diagram

$$\begin{tikzcd}
\mathcal{L}A \arrow[r, "\mathcal{L}\theta'_A"] \arrow[d, "\theta_{\mathcal{L}A}"'] & \mathcal{L}(up'(pf'A)) \arrow[r, "\rho_{pf'A}"] \arrow[d, "\theta_{\mathcal{L}(up'(pf'A))}"'] & up(\mathcal{T}(pf'A)) \arrow[d, "\theta_{up(\mathcal{T}(pf'A))}"'] \arrow[dr, bend left, "\mathrm{id}"] & \\
up(pf(\mathcal{L}H)) \arrow[r, "up(pf(\mathcal{L}\theta'_A))"] \arrow[rrr, bend right, "up\rho^\flat_A"'] & up(pf(\mathcal{L}(up'(pf'A)))) \arrow[r, "up(pf\rho_{pf'A})"] & up(pf(up(\mathcal{T}(pf'A)))) \arrow[r, "up\eta_{\mathcal{T}(pf'A)}"] & up(\mathcal{T}(pf'A))
\end{tikzcd}$$

Here the bottom square commutes by definition of $\rho^\flat$. The other two squares commute by naturality of θ and the triangle on the right commutes because θ and η are the units of a dual adjunction. $\square$

Proof of Proposition 3.27. Recall that $\theta'_{up'(X,\leq)}([\![\varphi]\!]^{\mathfrak{M}}) = \{\mathfrak{p} \in pf'(up'(X,\leq)) \mid [\![\varphi]\!]^{\mathfrak{M}} \in \mathfrak{p}\}$. So the first item is equivalent to

$$[\![\varphi]\!]^{\mathfrak{pe}_\tau \mathfrak{M}} = \theta'_{up'(X,\leq)}([\![\varphi]\!]^{\mathfrak{M}}),$$

where we view truth sets of formulae as elements in the relevant complex algebras (cf. Prop. 3.22). The proof proceeds by induction on the structure of φ. If $\varphi = q \in \mathrm{Prop}$ then the statement holds by definition of V^{pe}. The cases $\varphi = \top$ and $\varphi = \bot$ hold by definition of a prime filter.

If φ is of the form $\varphi_1 \star \varphi_2$, where $\star \in \{\wedge, \vee, \to\}$ then we use Lem. A to find

$$\begin{aligned}
\llbracket \varphi_1 \star \varphi_2 \rrbracket^{\mathfrak{pe}_\tau \mathfrak{M}} &= \llbracket \varphi_1 \rrbracket^{\mathfrak{pe}_\tau \mathfrak{M}} \star \llbracket \varphi_1 \rrbracket^{\mathfrak{pe}_\tau \mathfrak{M}} \\
&= \theta'_{\mathit{up}'(X,\leq)}(\llbracket \varphi_1 \rrbracket^{\mathfrak{M}}) \star \theta'_{\mathit{up}'(X,\leq)}(\llbracket \varphi_2 \rrbracket^{\mathfrak{M}}) && \text{(IH)} \\
&= \theta'_{\mathit{up}'(X,\leq)}(\llbracket \varphi_1 \rrbracket^{\mathfrak{M}} \star \llbracket \varphi_2 \rrbracket^{\mathfrak{M}}) \\
&= \theta'_{\mathit{up}'(X,\leq)}(\llbracket \varphi_1 \star \varphi_2 \rrbracket^{\mathfrak{M}})
\end{aligned}$$

The case where $\varphi = \heartsuit^\lambda(\varphi_1, \ldots, \varphi_n)$ follows from a similar computation, using the fact that $\theta'_{\mathit{up}(X,\leq)}$ preserves operators of the form $\heartsuit^\lambda$.

Item (ii) follows from Item (i) and the definition of $\eta_{(X,\leq)}(x)$ via

$$\mathfrak{M}, x \Vdash \varphi \quad \text{iff} \quad x \in \llbracket \varphi \rrbracket^{\mathfrak{M}} \quad \text{iff} \quad \llbracket \varphi \rrbracket^{\mathfrak{M}} \in \eta_{(X,\leq)}(x) \quad \text{iff} \quad \mathfrak{pe}_\tau \mathfrak{M}, \eta_{(X,\leq)}(x) \Vdash \varphi.$$

For Item (iii), let V be any valuation for $\mathfrak{X}$ and $x \in X$. By assumption $(\mathfrak{pe}_\tau \mathfrak{X}, V^{pe}), \theta'_{(X,\leq)}(x) \Vdash \varphi$, so by Item (ii) $(\mathfrak{X}, V), x \Vdash \varphi$ and hence $\mathfrak{X} \Vdash \varphi$. $\square$

Proof of Lemma 3.29. Let A be a Heyting algebra. Recall that $\theta'_A(a) = \{\mathfrak{q} \in \mathit{pf}'A \mid a \in \mathfrak{q}\}$. Using this we can rewrite $\tau^\Box_A : \mathit{pf}(\mathcal{L}^\Box A) \to \mathcal{P}_{\mathit{up}}(\mathit{pf}'A)$ as

$$\tau^\Box_A(Q) = \bigcap \{\theta'_A(a) \mid a \in A, \boxdot a \in Q\}. \qquad (\star)$$

Since $\theta'_A(a)$ is an upset of $\mathit{pf}'A$, $\tau^\Box_A(Q)$ is also an upset of $\mathit{pf}'A$, hence in $\mathcal{P}_{\mathit{up}}(\mathit{pf}'A)$. The elements of $\mathit{pf}(\mathcal{L}^\Box A)$ are ordered by inclusion. If $Q, Q' \in \mathit{pf}(\mathcal{L}^\Box A)$ and $Q \subseteq Q'$ then it follows immediately that $\tau^\Box_A(Q) \supseteq \tau^\Box_A(Q')$. Since $\mathcal{P}_{\mathit{up}}(\mathit{pf}'A)$ is ordered by reverse inclusion, so $\tau^\Box_A$ is a morphism of Pos.

For naturality, let $h : A \to B$ be a Heyting homomorphism. We need that

$$\begin{array}{ccc}
\mathit{pf}(\mathcal{L}^\Box A) & \xrightarrow{\tau^\Box_A} & \mathcal{P}_{\mathit{up}}(\mathit{pf}'A) \\
{\scriptstyle (\mathcal{L}^\Box h)^{-1}} \uparrow & & \uparrow {\scriptstyle \mathcal{P}_{\mathit{up}}(h^{-1})} \\
\mathit{pf}(\mathcal{L}^\Box B) & \xrightarrow{\tau^\Box_B} & \mathcal{P}_{\mathit{up}}(\mathit{pf}'B)
\end{array}$$

commutes. Let $Q \in \mathit{pf}(\mathcal{L}^\Box B)$, $\mathfrak{q} \in \mathit{pf}'A$, and suppose $\mathfrak{q} \in \tau^\Box_A(\mathcal{L}^\Box h)^{-1}(Q)$. To show $\mathfrak{q} \in \mathcal{P}_{\mathit{up}}(h^{-1})(\tau^\Box_B(Q))$ it suffices to find a prime filter $\mathfrak{p} \in \tau^\Box_B(Q)$ such that $h^{-1}(\mathfrak{p}) \subseteq \mathfrak{q}$, because $\tau^\Box_A(\mathcal{L}^\Box h)^{-1}(Q)$ is an upset of $\mathit{pf}'A$. Define $F = \{b \in B \mid \boxdot b \in Q\}$ and $I = \{b \in B \mid \exists a \in A \setminus \mathfrak{q} \text{ s.t. } b \leq c\}$. If $I \cap F \neq \emptyset$ then there exists $b \in B$ and $a \in A \setminus \mathfrak{q}$ such that $b \leq h(a)$. Since $\boxdot b \in Q$ this implies $\boxdot h(a) \in Q$ and hence $\boxdot a \in (\mathcal{L}^\Box h)^{-1}(Q)$. But then $a \in \mathfrak{q}$ because $\mathfrak{q} \in \tau^\Box_A((\mathcal{L}^\Box h)^{-1}(Q))$, a contradiction. So $I \cap F = \emptyset$. The prime filter lemma then gives a prime filter $\mathfrak{p}$ containing F and disjoint from I. This satisfies $\mathfrak{p} \in \tau^\Box_B(Q)$ and $h^{-1}(\mathfrak{p}) \subseteq \mathfrak{q}$ by design.

Conversely, suppose $\mathfrak{q} \in \mathcal{P}_{\mathit{up}}(h^{-1})(\tau^\Box_B(Q))$. Then there exists a $\mathfrak{p} \in \tau^\Box_B(Q)$ such that $\mathfrak{q} = h^{-1}(\mathfrak{p})$. We show that $\mathfrak{q} \in \tau^\Box_A(\mathcal{L}^\Box h)^{-1}(Q)$. Let $a \in A$ and suppose $\boxdot a \in (\mathcal{L}^\Box h)^{-1}(Q)$. Then $\boxdot h(a) = \mathcal{L}^\Box h(\boxdot a) \in Q$ so $h(a) \in \mathfrak{p}$. But this implies $a \in \mathfrak{q} = h^{-1}(\mathfrak{p})$. So by definition $\mathfrak{q} \in \tau^\Box_A(\mathcal{L}^\Box h)^{-1}(Q)$.

Finally, we show that $(\rho_A^\square)^\flat \circ \tau_A^\square = id_{pf \circ \mathcal{L}^\square A}$ for $A \in \mathsf{HA}$. Let $Q \in pf(\mathcal{L}^\square A)$. Since elements of $pf(\mathcal{L}^\square A)$ are determined uniquely by the generators of the form $\boxdot a$ they contain, it suffices to show that $\boxdot a \in Q$ iff $\boxdot a \in (\rho_A^\square)^\flat(\tau_A^\square(Q))$. Because of the computation in Exm. 3.28 this is equivalent to showing $\boxdot a \in Q$ iff $\tau_A^\square(Q) \subseteq \theta'_A(a)$. The direction from left to right follows from ($\star$). For the converse, suppose $\boxdot a \notin Q$. Let $F = \{b \in A \mid \square b \in Q\}$ and $I = \{c \in A \mid c \leq a\}$. Then F is a filter and I is an ideal of A, and $F \cap I = \emptyset$. By the prime filter lemma we obtain some $\mathfrak{q} \in pf'A$ extending F and disjoint from I. This implies that $\mathfrak{q} \in \tau_A^\square(Q)$ while $\mathfrak{q} \notin \theta'_A(a)$, so that $\tau_A^\square(Q) \not\subseteq \theta'_A(a)$. $\square$

Proof of Lemma 4.8. Throughout this proof use the fact that $pf'A$ forms an Esakia space (which in particular is a Stone space), with a topology generated by sets of the form $\theta'_A(a)$ and their complements [2, Sec. 2.3.3]. Furthermore, we note that for any Heyting homomorphism $h : A \to B$ we have

$$\theta'_B(h(a)) = (h^{-1})^{-1}(\theta'_A(a)) \tag{$\dagger$}$$

We first prove naturality of $\tau^\triangle$. Let $h : A \to B$ be a Heyting homomorphism. We need to show that the following diagram commutes:

$$\begin{array}{ccc} pf(\mathcal{L}^\triangle A) & \xrightarrow{\tau_A^\triangle} & \mathcal{P}_{up}(pf'A) \\ {\scriptstyle (\mathcal{L}^\triangle h)^{-1}}\uparrow & & \uparrow{\scriptstyle \mathcal{P}_{up}(h^{-1})} \\ pf(\mathcal{L}^\triangle B) & \xrightarrow{\tau_B^\triangle} & \mathcal{P}_{up}(pf'B) \end{array}$$

Let $Q \in pf(\mathcal{L}^\triangle B)$ and $D \in \mathcal{U}p(pf'A)$. We go by the items of Def. 4.7.

- If $D = \theta'_A(a)$ for some $a \in A$ then

$$\begin{array}{llll} \theta'_A(a) \in \tau_A^\triangle(\mathcal{L}^\triangle h)^{-1}(Q) & \text{iff} & \mathbin{\triangle\!\!\!\cdot}\, a \in (\mathcal{L}^\triangle h)^{-1}(Q) & (\text{Def. } \tau^\triangle) \\ & \text{iff} & (\mathcal{L}^\triangle h)(\mathbin{\triangle\!\!\!\cdot}\, a) \in Q & \\ & \text{iff} & \mathbin{\triangle\!\!\!\cdot}\, h(a) \in Q & (\text{Def. of } \mathcal{L}^\triangle) \\ & \text{iff} & \theta'_B(h(a)) \in \tau_B^\triangle(Q) & (\text{Def. of } \tau^\triangle) \\ & \text{iff} & (h^{-1})^{-1}(\theta'_A(a)) \in \tau_B^\triangle(Q) & (\text{By } (\dagger)) \\ & \text{iff} & \theta'_A(a) \in \mathcal{M}(h^{-1})(\tau_B^\triangle(Q)) & (\text{Def. of } \mathcal{M}) \end{array}$$

- Suppose D is closed in $pf'A$. If $D \in \tau_A^\triangle(\mathcal{L}^\triangle h)^{-1}(Q)$, then for all $a \in A$, $D \subseteq \theta'_A(a)$ implies $\mathbin{\triangle\!\!\!\cdot}\, a \in (\mathcal{L}^\triangle h)^{-1}(Q)$, i.e. $\mathbin{\triangle\!\!\!\cdot}\, h(a) \in Q$. In order to prove that $D \in \mathcal{M}(h^{-1})(\tau_B^\triangle(Q))$, we need to show that $(h^{-1})^{-1}(D) \in \tau_B^\triangle$. Since h^{-1} is an Esakia morphism (hence continuous), $(h^{-1})^{-1}(D)$ is closed in $pf'B$, so it suffices to show that for all $b \in B$, $(h^{-1})^{-1}(D) \subseteq \theta'_B(b)$ implies $\mathbin{\triangle\!\!\!\cdot}\, b \in Q$. Let $b \in B$ be such that $(h^{-1})^{-1}(D) \subseteq \theta'_B(b)$. Then since D is closed we have

$$\bigcap \big\{(h^{-1})^{-1}(\theta'_A(a)) \mid a \in A, D \subseteq \theta'_A(a)\} \subseteq \theta'_B(b)\big\}.$$

  Using ($\dagger$) and compactness of $pf'B$ we can find $a_1, \dots, a_n \in A$ such that

$$\theta'_B(h(a_1 \wedge \cdots \wedge a_n)) = \theta'_B(h(a_1)) \cap \cdots \cap \theta'_B(h(a_n)) \subseteq \theta'_B(b).$$

As a consequence of Esakia duality it follows that $h(a_1 \wedge \cdots \wedge a_n) \leq b$. Since $D \subseteq \theta'_A(a_1 \wedge \cdots \wedge a_n)$, we have $\vartriangle(a_1 \wedge \cdots \wedge a_n) \in (\mathcal{L}^{\vartriangle} h)^{-1}(Q)$, so $\vartriangle(h(a_1 \wedge \cdots \wedge a_n)) \in Q$. Monotonicity of $\vartriangle$ now implies $\vartriangle b \in Q$.

Conversely, if $D \in \mathcal{M}(h^{-1})(\tau_B^{\vartriangle}(Q))$ then a similar but easier argument shows that $D \in \tau_A^{\vartriangle}(\mathcal{L}^{\vartriangle} h)^{-1}(Q)$.

- Finally, suppose D is any upset. If $D \in \tau_A^{\vartriangle}(\mathcal{L}^{\vartriangle} h)^{-1}(Q)$ then there exists a closed upset C such that $C \subseteq D$ and $C \in \tau_A^{\vartriangle}(\mathcal{L}^{\vartriangle} h)^{-1}(Q)$. This implies $C \in \mathcal{M}(h^{-1})(\tau_B^{\vartriangle}(Q))$, so that $(h^{-1})^{-1}(C) \in \tau_B^{\vartriangle}(Q)$. Since $(h^{-1})^{-1}(C)$ is closed again and $(h^{-1})^{-1}(C) \subseteq (h^{-1})^{-1}(D)$ we have $(h^{-1})^{-1}(D) \in \tau_B^{\vartriangle}(Q)$, and therefore $D \in \mathcal{M}(h^{-1})(\tau_B^{\vartriangle}(Q))$.

  Conversely, suppose $D \in \mathcal{M}(h^{-1})(\tau_B^{\vartriangle}(Q))$. Then there exists a closed upset $C \in \tau_B^{\vartriangle}(Q)$ such that $C \subseteq (h^{-1})^{-1}(D)$. Define $C' = h^{-1}[C]$ to be direct image of C under h^{-1}. Since h^{-1} is an Esakia morphism it sends closed upsets to closed upsets. Furthermore $C \subseteq (h^{-1})^{-1}(C')$ so $C' \in \mathcal{M}(h^{-1})(\tau_B^{\vartriangle}(Q))$. This implies $C' \in \tau_A^{\vartriangle}(\mathcal{L}^{\vartriangle} h)^{-1}(Q)$. By design $C' \subseteq D$, hence $D \in \tau_A^{\vartriangle}(\mathcal{L}^{\vartriangle} h)^{-1}(Q)$.

Next we prove that $(\rho^{\vartriangle})^{\flat}_A \circ \tau_A = id_{pf(\mathcal{L}^{\vartriangle} A)}$ for $A \in \mathsf{HA}$. It follows from the definitions of $\rho^{\flat}$ and τ that for any Heyting algebra A, $a \in A$ and prime filter $Q \in pf(\mathcal{L}^{\vartriangle} A)$ we have $\theta'_A(a) \in \rho^{\flat}_A(\tau_A(Q))$ iff $\vartriangle a \in \tau_A(Q)$ iff $\theta'_A(a) \in Q$. Since elements of $pf(\mathcal{L}^{\vartriangle} A)$ are determined uniquely by the elements of the form $\vartriangle a$ they contain, this proves the lemma. $\square$