Completeness for an Intuitionistic Modal Logic of Vagueness

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Abstract

Wright has long advocated for an intuitionistic solution to the Sorites paradox. Recently, Bobzien and Rumfitt have suggested an extension to this solution by introducing the modality 'it is borderline whether', in part intended to provide the intuitionist with alternatives to assenting, dissenting, and remaining silent when asked questions about vague predicates (e.g., 'Is this tube red?'). Their proposal includes a collection of formulas and inference rules that they argue an intuitionistic modal logic of vagueness ought to prove. This paper proposes a logic meeting Bobzien and Rumfitt's desiderata, establishes a semantics for which the logic is sound and complete, and then uses completeness to prove a metatheorem asserting the equivalence of three notions of when the logic settles the matter of some formula. We then consider the addition of an axiom ruling out clear borderline cases, which is endorsed by proponents of columnar vagueness like Bobzien. Leaning heavily on a topological analogy, we show that the semantics can be adapted to accommodate this extension of the logic (and the corresponding view on higher-order vagueness) without losing completeness.

Keywords: Intutionism, intuitionistic modal logic, vagueness, completeness

1 Introduction

Bobzien and Rumfitt [3] defend Wright's [8] proposal to use the intuitionistic propositional calculus when reasoning about vague statements. If we were to present the intuitionist with an array of one hundred tubes whose colors imperceptibly shift, from the first tube to the last, from red to orange, she would not be obligated to accept $\mathrm{R}a_n \vee \neg\mathrm{R}a_n$ for each n, where R is a predicate for redness and a_n refers to the nth tube. Hence, even though the first tube is clearly red and the last is clearly not, the intuitionist does not find herself in the classicist's predicament of being forced to hold that, while each pair of consecutive tubes is indiscriminable, there exists a consecutive pair of tubes where the first of the pair is red and the second is not. As Bobzien and Rumfitt explain, "An intuitionist like Wright is unwilling to assert certain instances of the Law [of Excluded Middle], such as $\mathrm{R}a_{50} \vee \neg\mathrm{R}a_{50}$ with a_{50} supposed to be a borderline case of R"[3, p. 237].

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This picture misses something, however. When a tube of questionable redness is presented to the intuitionist, her silence is not an indication of her having nothing to say on the matter. Bobzien and Rumfitt write that in cases where "[the intuitionist] does not assert '$\mathrm{R}a_{50} \vee \neg \mathrm{R}a_{50}$'... she may invoke borderlineness" [3, p. 240]. Thus, there is a pull to extend the language so that it can express as much. Bobzien and Rumfitt argue for a number of principles that should govern a borderlineness modality ∇. Adding ∇ to the language of propositional logic, they take ∇A to mean that it is *borderline whether* A. The modalities $\Box$ and $\Diamond$ are then defined as $\Box A \equiv A \wedge \neg \nabla A$ and $\Diamond A \equiv A \vee \nabla A$. $\Box A$ can be taken to mean *it is clear that* A, while $\Diamond A$ can be taken to mean *it cannot be ruled out that* A [3, p. 242].

A number of axioms are suggested by Bobzien and Rumfitt, though they leave open whether this list is complete. In this paper, we will propose an intuitionistic modal logic of vagueness that strengthens one of their axioms, provide a formal semantics for their language, and prove the corresponding soundness and completeness theorems. These results will then be used to prove a metatheorem for our logic that establishes an equivalence among three candidate notions of a logic settling the matter of a formula φ.

For convenience, in what follows we will take $\Box$ and $\Diamond$ to be the primitive modalities, though this is of no material difference, since the axioms proposed by Bobzien and Rumfitt are strong enough to define ∇ in terms of $\Box$ and $\Diamond$. In particular, any system that they endorse will have $\vdash \nabla\varphi \leftrightarrow (\Diamond\varphi \wedge \neg\Box\varphi)$. After fixing a countably infinite set $\mathbf{P} = \{p, q, \ldots\}$ of propositional variables, we define the language $\mathcal{L}$ recursively, as follows:

$$\varphi ::= \bot \,|\, p \,|\, (\varphi \wedge \varphi) \,|\, (\varphi \vee \varphi) \,|\, (\varphi \to \varphi) \,|\, \Box\varphi \,|\, \Diamond\varphi$$

where $p \in \mathbf{P}$. $\mathcal{L}_0$ will denote the subset of modal-free formulas—those formulas with no occurrences of $\Box$ or $\Diamond$. We take $\neg\varphi$, $\top$, and $\varphi \leftrightarrow \psi$ to be shorthands for $\varphi \to \bot$, $\bot \to \bot$, and $(\varphi \to \psi) \wedge (\psi \to \varphi)$, respectively. The axioms amassed by Bobzien and Rumfitt are the axioms in Figure 1 appearing above the second dashed line, as well as the *stable nabla* axiom $\mathbf{S}\nabla$ $(\neg\neg\nabla p \to \nabla p)$. [2]

For the logic of vagueness proposed in this paper, we first argue that $\mathbf{S}\nabla$ ought to be strengthened to $\neg\neg\Diamond p \to \Diamond p$ ($\mathbf{S}\Diamond$). Because we have an equivalence between ∇p and $\Diamond p \wedge \neg\Box p$, the antecedent of $\mathbf{S}\nabla$ can be written as $\neg\neg(\Diamond p \wedge \neg\Box p)$, which is in turn equivalent to $\neg\neg\Diamond p \wedge \neg\Box p$ by an intuitionistically acceptable argument. The consequent of $\mathbf{S}\nabla$ is similarly equivalent to $\Diamond p \wedge \neg\Box p$, so we can equivalently express the axiom as $(\neg\neg\Diamond p \wedge \neg\Box p) \to \Diamond p$. But now it seems that we may as well strengthen the axiom by dropping $\neg\Box p$ from the antecedent. After all, if we are trying to capture conditions sufficient for concluding *it cannot be ruled out that* p, why would knowing that p is not clearly true help our case?

[2] It should be noted that the first Fischer Servi axiom **FS1** is mentioned but not defended in [3]. We will leave the matter unsettled as they did.

This brings us to the system IVL (intuitionistic vagueness logic), which comprises the aforementioned axioms along with three uncontroversial deduction rules. For the rest of this paper, we will write $\vdash$ for $\vdash_{\mathsf{IVL}}$. We present IVL in Figure 1, where the Fischer Servi logic FS can be obtained by restricting to the axioms above the first dashed line (but keeping all of the inference rules).

The System IVL	
I	any theorem of IPC
K□a	$\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$
K□b	$\Box\top$
K◊a	$\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$
K◊b	$\neg\Diamond\bot$
FS1	$(\Diamond p \to \Box q) \to \Box(p \to q)$
FS2	$\Diamond(p \to q) \to (\Box p \to \Diamond q)$
T□	$\Box p \to p$
T◊	$p \to \Diamond p$
4□	$\Box p \to \Box\Box p$
4◊	$\Diamond\Diamond p \to \Diamond p$
S◊	$\neg\neg\Diamond p \to \Diamond p$
`MP`	from $\varphi \to \psi$ and φ infer ψ
`US`	from φ infer $\varphi[\psi/p]$
`Reg`	from $\varphi \to \psi$ infer $\bigcirc\varphi \to \bigcirc\psi$

Fig. 1. φ and ψ are any formulas in $\mathcal{L}$ and $\bigcirc \in \{\Box, \Diamond\}$.

When we present deductions, we will make free use of intuitionistic reasoning without spelling out every line. Such moves are, of course, just a number of instances of **I** and applications of `MP` and `US`. Where possible, we will give deductions in the weaker system FS to highlight which theorems do not depend on any $\mathsf{S4}$-like properties. We will now offer deductions of a couple well-known theorems of FS, both of which will be of use to us in later sections.

Lemma 1.1 $\vdash_{\mathsf{FS}} (\Box p \wedge \Diamond q) \to \Diamond(p \wedge q)$.

Proof.

$$\vdash_{\mathsf{FS}} q \to (p \to p \wedge q) \qquad (1)$$
$$\vdash_{\mathsf{FS}} \Diamond q \to \Diamond(p \to p \wedge q) \qquad \texttt{Reg}, (1) \qquad (2)$$
$$\vdash_{\mathsf{FS}} \Diamond q \to (\Box p \to \Diamond(p \wedge q)) \qquad \mathbf{FS2}, (2) \qquad (3)$$
$$\vdash_{\mathsf{FS}} (\Box p \wedge \Diamond q) \to \Diamond(p \wedge q) \qquad (3) \qquad (4)$$

□

Proposition 1.2 $\vdash_{\mathsf{FS}} \neg\Diamond p \leftrightarrow \Box\neg p$.

Proof.

$\vdash_{\mathsf{FS}} \neg\Diamond p \to (\Diamond p \to \Box\bot)$		(1)
$\vdash_{\mathsf{FS}} \neg\Diamond p \to \Box(p \to \bot)$	**FS1**, (1)	(2)
$\vdash_{\mathsf{FS}} \neg\Diamond p \to \Box\neg p$	(2)	(3)
$\vdash_{\mathsf{FS}} (\Box\neg p \wedge \Diamond p) \to \Diamond(p \wedge \neg p)$	Lemma 1.1	(4)
$\vdash_{\mathsf{FS}} (\Box\neg p \wedge \Diamond p) \to \Diamond\bot$	(4)	(5)
$\vdash_{\mathsf{FS}} \neg\Diamond\bot$	$\mathbf{K}\Diamond\mathbf{b}$	(6)
$\vdash_{\mathsf{FS}} \Box\neg p \to \neg\Diamond p$	(5), (6)	(7)
$\vdash_{\mathsf{FS}} \neg\Diamond p \leftrightarrow \Box\neg p$	(3), (7)	(8)

□

2 Semantics

2.1 Relational Preliminaries

As the formal semantics we develop will be relational, we will first establish some conventions for relations and operations on them. A relation R on a set A is a subset of A^2, and we write $a\ R\ b$ if $(a, b) \in R$. If R is a relation on A, we define $R^{-1} = \{(a, b) \in A^2 : b\ R\ a\}$. We also have a notion of relation composition. For two relations R_1 and R_2 on A, we set

$$R_1 \circ R_2 = \{(a, b) \in A^2 : \text{there exists } x \text{ such that } a\ R_2\ x \text{ and } x\ R_1\ b\}.$$

If $a \in A$ and R is a relation on A, we set $R\ (a) = \{b \in A : a\ R\ b\}$. In a few places, we will need to appeal to the transitive closure R^* of a relation R, which is the smallest transitive relation extending R.

Turning to partial orders specifically, if $(A, \preccurlyeq)$ is a poset, we denote by $\mathrm{Up}(A, \preccurlyeq)$ the collection of all upwardly closed subsets (up-sets) of A. For any $B \subseteq A$, we denote the upward closure of B as $\uparrow(B)$. The *principal up-set generated by* a is just $\uparrow(\{a\})$, which we will shorten to $\uparrow(a)$ when it is clear that a is an element of the underlying set of the partial order in question. Finally, for $C \subseteq B \subseteq A$, we say that C is *cofinal* in B if for every $b \in B$, there exists $c \in C$ with $b \preccurlyeq c$.

2.2 Frames

The semantics proposed is the same as for the Fischer Servi logic FS, restricting to a smaller class of frames to accommodate the additional axioms.

Definition 2.1 *A Fischer Servi frame is a triple* $(W, \preccurlyeq, R)$ *where* $\preccurlyeq$ *is a partial order and* R *is a binary relation that satisfy the following conditions:*

(FC1) $(\preccurlyeq \circ R) \subseteq (R \circ \preccurlyeq)$;
(FC2) $(\preccurlyeq \circ R^{-1}) \subseteq (R^{-1} \circ \preccurlyeq)$.

A Fischer Servi model is a Fischer Servi frame equipped with a function

$v : \mathbf{P} \to \mathrm{Up}(W, \preccurlyeq)$. For such a model $\mathcal{M} = (W, \preccurlyeq, R, v)$, the semantics follows:

$$\begin{array}{lll}
\mathcal{M}, w \nVdash \bot & & \\
\mathcal{M}, w \Vdash p & \text{iff} & w \in v(p) \\
\mathcal{M}, w \Vdash (\varphi \wedge \psi) & \text{iff} & \mathcal{M}, w \Vdash \varphi \text{ and } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash (\varphi \vee \psi) & \text{iff} & \mathcal{M}, w \Vdash \varphi \text{ or } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash (\varphi \to \psi) & \text{iff} & \text{for all } x \succcurlyeq w,\ \mathcal{M}, x \nVdash \varphi \text{ or } \mathcal{M}, x \Vdash \psi \\
\mathcal{M}, w \Vdash \Box\varphi & \text{iff} & \text{for all } x \in (R \circ \preccurlyeq)(w),\ \mathcal{M}, x \Vdash \varphi \\
\mathcal{M}, w \Vdash \Diamond\varphi & \text{iff} & \text{there exists } x \in R\ (w) \text{ such that } \mathcal{M}, x \Vdash \varphi.
\end{array}$$

Fischer Servi studied extensions of the logic FS using the above semantics in, for example, [4] and [5], with one of her foundational results in the study of such logics being the completeness of FS with respect to the class of Fischer Servi frames. Looking at the forcing clauses, the relation $\preccurlyeq$ plays the roll of the partial order that appears in Kripke models for IPC and the relation R acts similarly to the relation that appears in Kripke models for classical normal modal logics. Accordingly, we might refer to $\preccurlyeq$ as the *intuitionistic relation* and R as the *modal relation.* As it stands, our class of models is too large to obtain a soundness result, so we will have to make further demands on frames.

Definition 2.2 *An* S4 *Fischer Servi frame is a Fischer Servi frame* $(W, \preccurlyeq, R)$ *where* R *is moreover a quasi-order.*

We call this frame class $\mathcal{S}$. It is already known that the logic $\mathsf{FSS4} = \mathsf{FS} \oplus \{\mathbf{T}\Diamond, \mathbf{T}\Box, \mathbf{4}\Box, \mathbf{4}\Diamond\}$ is complete with respect to this class [1].

At this point, it will be useful to introduce diagrams, which may provide better intuition than the relation composition notation used thus far. For this paper, we will use the convention that single-line arrows indicate the intuitionistic relation and double-line arrows indicate the modal relation. (Where possible, the intuitionistic arrows will point up and the modal arrows will point to the side.) From this point on, we are concerned with only birelational structures where both relations are quasi-orders, so we can and will unambiguously omit all self-loops as well as arrows whose existences are implied by transitivity. Finally, dotted arrows and hollow points are used to mark existential quantifiers and instances of relations appearing in the consequent. Figure 2 illustrates conditions (FC1) and (FC2).

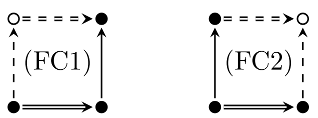


Fig. 2. The Fischer Servi frame conditions from Definition 2.1

To obtain IVL from FSS4 we need only extend by the axiom $\mathbf{S}\Diamond$. Analogously, we need just one additional imposition on our frame class to obtain a soundness result. Consider the following first-order frame condition:

(FC3-weak)	For every $w \in W$, there exists $w' \succcurlyeq w$ such that $(R \circ \preccurlyeq)(w') \subseteq (\succcurlyeq \circ R)(w)$.

Perhaps easier to understand is the diagram in Figure 3.

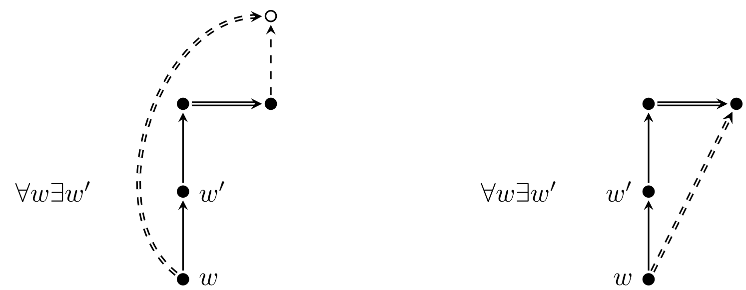


Fig. 3. The diagrams for conditions (FC3-weak) (left) and (FC3) (right)

Proposition 2.3 (Correspondence) *A frame in $\mathcal{S}$ validates* $\mathbf{S}\Diamond$ *if and only if it satisfies* (FC3-weak)*.*

Proof. It is straightforward to check that any S4 Fischer Servi frame satisfying (FC3-weak) will force $\neg\neg\Diamond p \to \Diamond p$ at every point.

We will now check the other direction. For convenience, let $\Phi(w, x, y)$ abbreviate the first-order property $(x\ R \circ \preccurlyeq\ y)$ and $\neg(\exists y')(y' \succcurlyeq y$ and $w\ R\ y')$. Suppose that some S4 Fischer Servi frame $\mathcal{F} = (W, \preccurlyeq, R)$ fails to satisfy (FC3-weak). Then there is a point $w \in W$ such that set $A = \{x \in W : (\exists y)\Phi(w, x, y)\}$ is cofinal in the principal up-set of w. For each element $x \in A \cap {\uparrow}(w)$ choose a $y(x)$ witnessing $(\exists y)\Phi(w, x, y)$, and fix a valuation with $v(p) = {\uparrow}(\{y(x) : x \in A \cap {\uparrow}(w)\})$. This will yield $w \Vdash \neg\neg\Diamond p$, as the points above w that have a modal successor forcing p are cofinal in ${\uparrow}(w)$. On the other hand, $w \nVdash \Diamond p$, as forcing $\Diamond p$ is equivalent to having a modal successor that is an intuitionistic successor of some $y(x)$, and each $y(x)$ was picked so that this is impossible. □

As it turns out, we can actually get away with a slightly stronger—and easier to work with—condition than (FC3-weak) without losing completeness, so we will end up taking our class of frames to be slightly smaller than the class of all S4 Fischer Servi frames validating $\mathbf{S}\Diamond$.

(FC3) For every $w \in W$, there exists $w' \succcurlyeq w$ such that $(R \circ \preccurlyeq)(w') \subseteq R\ (w)$.

We will sometimes refer to the point w' in condition (FC3) as a *diamond-reflection point for w*. (FC3) can then be thought of as simply asserting that every point w has a diamond-reflection point.

Definition 2.4 *An* intuitionistic vagueness *(IV) frame is an* S4 *Fischer Servi frame satisfying the condition (FC3).*

We call the class of IV frames $\mathcal{V}$.

2.3 Intuitionistic Vagueness Models

An IV model is an IV frame equipped with a function $v : \mathbf{P} \to \mathrm{Up}(W, \preccurlyeq)$.

Proposition 2.5 *For any IV model $\mathcal{M} = (W, \preccurlyeq, R, v)$, $\varphi \in \mathcal{L}$, and $w \preccurlyeq w'$, if $w \Vdash \varphi$ then $w' \Vdash \varphi$.*

Proof. The proof is by induction on formula complexity. The argument for each non-modal case is the standard one given for intuitionistic logic. The $\Box$ case is built into the semantics. For the $\Diamond$ case, suppose that $w \Vdash \Diamond\psi$. Then there is some $x \in W$ with $w\ R\ x$ and $x \Vdash \psi$. By frame condition (FC2), there is an $x' \succcurlyeq x$ with $w'\ R\ x'$. By the induction hypothesis, $x' \Vdash \psi$, so $w' \Vdash \Diamond\psi$. □

Theorem 2.6 (Soundness) IVL *is sound with respect to* $\mathcal{V}$.

Proof. We will just check that $\mathbf{S}\Diamond$ holds at every point in every IV model, as all of the other axioms and the rules are not novel to this paper. Fix a model $\mathcal{M} = (W, \preccurlyeq, R, v)$ and suppose that $w \Vdash \neg\neg\Diamond\varphi$ for some $w \in W$ and $\varphi \in \mathcal{L}$. By condition (FC3), we can find a $w' \succcurlyeq w$ such that whenever there is a point x with $w' \preccurlyeq w''\ R\ x$, it is the case that $w\ R\ x$. There must be a point $w''' \succcurlyeq w'$ with $w''' \vDash \Diamond\varphi$. w''' sees a point x forcing φ, but by condition (FC3), w also sees x. We conclude $w \Vdash \Diamond\varphi$. □

The argument above sheds light on the term *diamond-reflection.* In any model where w' is a diamond-reflection point of w, $w \Vdash \Diamond\varphi$ if and only if $w' \Vdash \Diamond\varphi$, for every formula φ. Further, since any successor of w' is also a diamond-reflection point of w, we have $w \nVdash \Diamond\varphi$ if and only if $w' \Vdash \neg\Diamond\varphi$.

The soundness theorem allows us to quickly verify that IVL is not, in a few senses, too strong. *A priori* it seems possible that the underlying propositional calculus of our system is stronger than IPC via some deduction making use of the modalities. Another concern is the status of $\Box p \vee \neg\Box p$. Bobzien and Rumfitt begin their exploration by extending the basic propositional language by $\Box$. Now, the classicist can deny $\Box p \vee \Box\neg p$; even classically, some propositions are neither clearly the case nor clearly not the case. This does not solve the problem of vagueness for them, however, as this move alone still demands commitment to $\Box p \vee \neg\Box p$.

We can give a very simple argument to address the first concern.

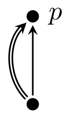
Corollary 2.7 IVL *is a conservative extension of* IPC*.*

Proof. Let $\varphi \in \mathcal{L}_0$ be a non-theorem of IPC. By completeness of IPC [6], we can find some model $\mathcal{M} = (W, \preccurlyeq, v)$ with $\mathcal{M}, x \nVdash \varphi$ for some $x \in W$. Extend this to an intuitionistic vagueness model by setting $\mathcal{M}^* = (W, \preccurlyeq, W^2, v)$. Clearly, we have $\mathcal{M}^*, x \nVdash \varphi$. Hence by Proposition 2.6, φ is not a theorem of IVL. □

Turning to the second concern, we introduce the technique of drawing models, which is an efficient method of producing corollaries of soundness. Our conventions for these drawings will be largely the same as for frame condition diagrams. For arguments involving a certain subset of a model, dotted arrows and hollow points will be reserved for applications of the frame conditions (FC1) and (FC2), but all quantification should be made clear by the accompanying text. We will now employ this technique to dispel the second concern.

Corollary 2.8 IVL *does not prove* $\Box p \vee \neg\Box p$.

Proof. We present a countermodel.



The lower point forces neither $\Box p$ nor $\neg\Box p$. □

A surprising feature of IVL is that it does not have the disjunction property: there are formulas φ and ψ such that $\vdash \varphi \vee \psi$, but $\nvdash \varphi$ and $\nvdash \psi$.

Proposition 2.9 *For any φ, $\vdash \Diamond\varphi \vee \Diamond\neg\varphi$.*

Proof.

$\vdash \neg\neg(\varphi \vee \neg\varphi)$		(1)
$\vdash \Diamond\neg\neg(\varphi \vee \neg\varphi)$	$\mathbf{T}\Diamond$, (1)	(2)
$\vdash \neg\neg\Diamond\neg\neg(\varphi \vee \neg\varphi)$	(2)	(3)
$\vdash \neg\Box\neg\neg\neg(\varphi \vee \neg\varphi)$	Proposition 1.2, (3)	(4)
$\vdash \neg\Box\neg(\varphi \vee \neg\varphi)$	(4)	(5)
$\vdash \neg\neg\Diamond(\varphi \vee \neg\varphi)$	Proposition 1.2, (5)	(6)
$\vdash \Diamond(\varphi \vee \neg\varphi)$	$\mathbf{S}\Diamond$, (6)	(7)
$\vdash \Diamond\varphi \vee \Diamond\neg\varphi$	$\mathbf{K}\Diamond\mathbf{a}$, (7)	(8)

□

Corollary 2.10 IVL *does not have the disjunction property.*

Proof. By Proposition 2.9, $\vdash \Diamond p \vee \neg\Diamond p$. By the soundness theorem for IVL, we have both $\nvdash \Diamond p$ and $\nvdash \Diamond\neg p$ as there is an obvious one-point countermodel in each case. □

2.4 Model Constructions

IV models are decidedly less flexible than Fischer Servi models. For instance, given two Fischer Servi models, one can take their disjoint union and add a new universal intuitionistic predecessor that does not stand in the modal relation with any other points. It is straightforward to check that this is still a Fischer Servi model, which, alongside soundness and completeness, furnishes a neat argument for the disjunction property of the logic FS. In the case of FS extended by the S4 principles, the same argument works, with the small caveat that one lets the new point stand in the modal relation with itself. IVL does not have the disjunction property, however, so we know that when we consider the disjoint union of two IV models with a new universal predecessor, there can be no general procedure to choose the modal relation such that both the new structure is an IV model and the forcing relation is preserved at the old points. Two typical constructions are still available to us, however.

Definition 2.11 *For each i in some index set I, let $\mathcal{M}_i = (W_i, \preccurlyeq_i, R_i, v_i)$ be* *an IV model. We define their disjoint union as*

$$\bigsqcup_{i \in I} \mathcal{M}_i = \left(\bigsqcup_{i \in I} W_i, \bigsqcup_{i \in I} \preccurlyeq_i, \bigsqcup_{i \in I} R_i, \bigsqcup_{i \in I} v_i \right).$$

The following proposition is straightforward to verify:

Proposition 2.12 *Let $\{\mathcal{M}_i\}_{i \in I}$ be a collection of IV models. Then $\bigsqcup_{i \in I} \mathcal{M}_i$ is an IV model and for each $j \in I$, $w \in W_i$, and $\varphi \in \mathcal{L}$, $\mathcal{M}_j, w \Vdash \varphi$ if and only if $\bigsqcup_{i \in I} \mathcal{M}_i, w \Vdash \varphi$.*

Definition 2.13 *Let $\mathcal{M} = (W, \preccurlyeq, R, v)$ be an IV model with $w \in W$. Then the submodel generated by w is $\mathcal{M}(w)$ where we have replaced W with the image of w under $(\preccurlyeq \cup R)^*$ and restricted the relations and valuation to this set.*

Also easily proved is the natural analog of Proposition 2.12 for generated submodels, which we now state.

Proposition 2.14 *Let $\mathcal{M}$ be an IV model with $w \in W$. Then $\mathcal{M}(w)$ is an IV model, and for any formula φ and x in the universe of $\mathcal{M}(w)$, we have $\mathcal{M}, x \Vdash \varphi$ if and only $\mathcal{M}(w), x \Vdash \varphi$.*

Because of the Fischer Servi frame condition (FC1) and the transitivity of both the modal and intuitionistic relations in an IV model, we can also obtain a simple characterization of the universe of a generated submodel.

Proposition 2.15 *Let $\mathcal{M} = (W, \preccurlyeq, R, v)$ be an IV model with $w \in W$. Then $(\preccurlyeq \cup R)^*(w) = (R \circ \preccurlyeq)(w)$.*

Proof. It is clear that any point in $(R \circ \preccurlyeq)(w)$ is in the transitive closure of the union of $\preccurlyeq$ and R. In the other direction, suppose that $x \in (\preccurlyeq \cup R)^*(w)$. Then there is a sequence of points and relations $w \sim_1 x_1 \sim_2 x_2 \cdots x_{n-1} \sim_n x$, where each $\sim_i$ is either R or $\preccurlyeq$. By frame condition (FC1), whenever $\sim_i$ is R and $\sim_{i+1}$ is $\preccurlyeq$, we can replace x_i with some new point and swap the relations. Iterate this process until each instance of $\preccurlyeq$ occurs before each instance of R in the sequence. That is, we get some sequence $w \preccurlyeq y_1 \cdots y_{k-1} \preccurlyeq y_k \; R \; y_{k+1} \cdots y_{n-1} \; R \; x$. Note that w and x are left intact, as we only swap out points that occur between two relations in the sequence. By the transitivity of each relation, we can now contract our sequence, so that we simply have $w \preccurlyeq y_k \; R \; x$. In other words, $x \in (R \circ \preccurlyeq)(w)$. □

3 Completeness

3.1 The Completeness Theorem

We will prove completeness of IVL with respect to the semantics in §2.2 using a canonical model argument. First we will derive the traditional modal inference rule of necessitation in our system.

Lemma 3.1 *The rule* `Nec` *(from φ infer $\Box\varphi$) is derivable in* IVL.

Proof.

$\vdash \varphi$	assumption	(1)
$\vdash \top \to \varphi$	(1)	(2)
$\vdash \Box\top \to \Box\varphi$	$\texttt{Reg}, (2)$	(3)
$\vdash \Box\top$	$\mathbf{K\Box b}$	(4)
$\vdash \Box\varphi$	$\texttt{MP}, (3), (4)$	(5)

□

Next we define the theories over $\mathcal{L}$ that will serve as the points in our canonical model. This definition, and the shape of the completeness proof in general, is essentially just a modalized version of the standard proof for the completeness of IPC, as found, e.g., in [6].

Definition 3.2 $\Gamma \subseteq \mathcal{L}$ *is a prime theory if it is deductively closed: if* $\Gamma \vdash \varphi$, *then* $\varphi \in \Gamma$*; consistent:* $\Gamma \nvdash \bot$*; and disjunctive: if* $\varphi \vee \psi \in \Gamma$*, then* $\varphi \in \Gamma$ *or* $\psi \in \Gamma$.

Definition 3.3 *For* $\Gamma, \Delta \subseteq \mathcal{L}$, (Γ, Δ) *is a consistent pair if for every* $\{\varphi_i\}_{i=1}^n \subseteq \Gamma$ *and* $\{\psi_j\}_{j=1}^m \subseteq \Delta$*, we have* $\nvdash \wedge_{i=1}^n \varphi_i \to \vee_{j=1}^m \psi_i$.

Lemma 3.4 *If* (Γ, Δ) *is a consistent pair, then there is a prime theory* $\Gamma' \supseteq \Gamma$ *such that* $\Gamma' \cap \Delta = \varnothing$.

Proof. A standard recursive argument suffices. □

If Γ' is a prime theory and $(\Gamma', \mathcal{L} \setminus \Gamma')$ extends (Γ, Δ), we might say simply that the prime theory Γ' extends the pair (Γ, Δ) without making reference to $\mathcal{L} \setminus \Gamma'$.

For convenience, we fix a few pieces of notation for some set Γ of formulas: $\Gamma^{\Box} = \{\varphi \in \mathcal{L} : \Box\varphi \in \Gamma\}$; $\Gamma^{\Diamond} = \{\varphi \in \mathcal{L} : \Diamond\varphi \in \Gamma\}$; $B(\Gamma) = \{\Box\psi \in \mathcal{L} : \psi \in \Gamma\}$; $D(\Gamma) = \{\Diamond\psi \in \mathcal{L} : \psi \in \Gamma\}$; and $N(\Gamma) = \{\nabla\psi \in \mathcal{L} : \psi \in \Gamma\}$. These will significantly streamline the notation in the definition of the canonical model, as well as in the proofs of the completeness theorem and some of its corollaries.

Definition 3.5 *The canonical model for* IVL *is* $\mathcal{M}^{\mathsf{C}} = (W^{\mathsf{C}}, \preccurlyeq^{\mathsf{C}}, R^{\mathsf{C}}, v^{\mathsf{C}})$ *where:*

(a) $\mathcal{M}^{\mathsf{C}}$ *is the set of all prime theories over* IVL*;*

(b) $\Gamma \preccurlyeq^{\mathsf{C}} \Gamma'$ *if and only* $\Gamma \subseteq \Gamma'$*;*

(c) $\Gamma\ R^{\mathsf{C}}\ \Delta$ *if and only if* $\Gamma^{\Box} \subseteq \Delta \subseteq \Gamma^{\Diamond}$*;*

(d) $v^{\mathsf{C}}(p) = \{\Gamma \in W^{\mathsf{C}} : p \in \Gamma\}$.

Proposition 3.6 *The canonical model is an intuitionistic vagueness model.*

Proof. It is clear that $v^{\mathsf{C}}(p)$ is an upset for each p. It is also immediate that $\preccurlyeq^{\mathsf{C}}$ is a partial order, as it is just set containment.

First we check that R^{C} is a quasi-order. It is reflexive since $\Gamma^{\Box} \subseteq \Gamma \subseteq \Gamma^{\Diamond}$ by the deductive closure of Γ and the axioms $\mathbf{T\Diamond}$ and $\mathbf{T\Box}$. For transitivity, suppose that $\Gamma\ R^{\mathsf{C}}\ \Delta\ R^{\mathsf{C}}\ \Theta$. If $\Box\varphi \in \Gamma$ then $\Box\Box\varphi \in \Gamma$ by $\mathbf{4\Box}$, so $\Box\varphi \in \Delta$, so

$\varphi \in \Theta$. On the other hand, if $\varphi \in \Theta$, then $\Diamond\varphi \in \Delta$, so $\Diamond\Diamond\varphi \in \Gamma$, so $\Diamond\varphi \in \Gamma$ by **4**$\Diamond$.

Now we check the frame conditions:

(FC1) Suppose that $\Gamma \ R^{\mathsf{C}} \ \Delta \ \preccurlyeq^{\mathsf{C}} \ \Delta'$. We first argue that $(\Gamma \cup D(\Delta'), B(\mathcal{L} \setminus \Delta'))$ is a consistent pair. If it were not, then we would have $\Gamma \Vdash \wedge_{i=1}^{n}\Diamond\varphi_i \to \vee_{j=1}^{m}\Box\psi_j$, where $\varphi_i \in \Delta'$ and $\psi_j \notin \Delta'$. Since $\Vdash \Diamond \wedge_{i=1}^{n} \varphi_i \to \wedge_{i=1}^{n}\Diamond\varphi_i$ and $\Vdash \vee_{j=1}^{m}\Box\psi_j \to \Box \vee_{j=1}^{m} \psi_j$, we actually have the more usable

$$\Gamma \Vdash \Diamond \wedge_{i=1}^{n} \varphi_i \to \Box \vee_{j=1}^{m} \psi_j.$$

The conjunction in the antecedent is in Δ' by deductive closure, and the disjunction in the consequent is not in Δ' by disjunctivity, so we can rewrite this as $\Gamma \Vdash \Diamond\varphi \to \Box\psi$ with $\varphi \in \Delta$ and $\psi \notin \Delta$. By **FS1**, we then obtain $\Gamma \Vdash \Box(\varphi \to \psi)$, so $\varphi \to \psi \in \Delta$ and subsequently $\varphi \to \psi \in \Delta'$. By deductive closure, $\psi \in \Delta'$, which is a contradiction. Therefore, $(\Gamma \cup D(\Delta'), B(\mathcal{L} \setminus \Delta'))$ is a consistent pair. Using Propositions 3.4, we can find a prime theory Γ' extending this pair. By construction, $\Gamma \subseteq \Gamma'$, and $\Gamma'^{\Box} \subseteq \Delta' \subseteq \Gamma'^{\Diamond}$. We see that $\Gamma \preccurlyeq^{\mathsf{C}} \Gamma' \ R^{\mathsf{C}} \ \Delta'$, as desired.

(FC2) Now suppose that $\Gamma \preccurlyeq^{\mathsf{C}} \Gamma'$ and $\Gamma \ R^{\mathsf{C}} \ \Delta$. We want to find Δ' such that $\Delta \preccurlyeq^{\mathsf{C}} \Delta'$ and $\Gamma' \ R^{\mathsf{C}} \ \Delta'$. In this case, we want to check that $(\Delta \cup \Gamma'^{\Box}, \mathcal{L} \setminus \Gamma'^{\Diamond})$ is a consistent pair. If it were inconsistent, we would have $\Delta \Vdash \varphi \to \psi$ where $\varphi \in \Gamma'^{\Box}$ and $\psi \notin \Gamma'^{\Diamond}$. By the definition of R^{C}, $\Gamma \Vdash \Diamond(\varphi \to \psi)$. By **FS2**, $\Gamma \Vdash \Box\varphi \to \Diamond\psi$. Γ' inherits this, and since $\Box\varphi \in \Gamma'$, deductive closure ensures $\Gamma' \Vdash \Diamond\psi$, which is a contradiction. Any prime theory Δ' extending the pair in question will then witness this instance of condition (FC2).

(FC3) Let $\Gamma \in W^{\mathsf{C}}$. We first want to show that $\Gamma \cup \{\neg\Diamond\varphi \in \mathcal{L} : \Diamond\varphi \notin \Gamma\}$ is consistent, since any prime theory extending it will be a diamond-reflection point for Γ. If it is not consistent, then for some $\{\neg\Diamond\varphi_i\}_{i=1}^{n} \subseteq \{\neg\Diamond\varphi \in \mathcal{L} : \Diamond\varphi \notin \Gamma\}$, we have $\Gamma \vdash \neg \wedge_{i=1}^{n} \neg\Diamond\varphi_i$. We argue as follows:

$\Gamma \vdash \neg \wedge_{i=1}^{n} \neg\Diamond\varphi_i$	assumption	(1)
$\Gamma \vdash \neg\neg \vee_{i=1}^{n} \Diamond\varphi_i$	(1)	(2)
$\Gamma \Vdash \neg\neg\Diamond \vee_{i=1}^{n} \varphi_i$	**K**$\Diamond$**a**, (2)	(3)
$\Gamma \Vdash \Diamond \vee_{i=1}^{n} \varphi_i$	**S**$\Diamond$, (3)	(4)
$\Gamma \Vdash \vee_{i=1}^{n}\Diamond\varphi_i$	**K**$\Diamond$**a**, (4)	(5)
$\vee_{i=1}^{n} \Diamond\varphi_i \in \Gamma$	deductive closure, (5)	(6)
$\Diamond\varphi_k \in \Gamma$ for some $1 \leq k \leq n$	disjunctivity, (6)	(7)

This is contradiction, so we conclude that $\Gamma \cup \{\neg\Diamond\varphi \in \mathcal{L} : \Diamond\varphi \notin \Gamma\}$ is consistent and choose Γ' to be an element of W^{C} containing $\Gamma \cup \{\neg\Diamond\varphi \in \mathcal{L} : \Diamond\varphi \notin \Gamma\}$. We check $(\mathcal{M}^{\mathsf{C}} \circ \preccurlyeq^{\mathsf{C}})(\Gamma') \subseteq \mathcal{M}^{\mathsf{C}}(\Gamma)$. For any successor $\Gamma' \preccurlyeq^{\mathsf{C}} \Gamma''$ with $\Gamma'' \ R^{\mathsf{C}} \ \Delta$, we have $\Gamma^{\Box} \subseteq \Gamma''^{\Box} \subseteq \Delta$. On the other hand, if $\varphi \in \Delta$, we must have $\Diamond\varphi \in \Gamma$, since if not, $\neg\Diamond\varphi \in \Gamma''$, which would mean $\Delta \nsubseteq \Gamma''^{\Diamond}$.

□

Lemma 3.7 (Truth lemma) $\varphi \in \Gamma$ *if and only if* $\mathcal{M}^{\mathsf{C}}, \Gamma \Vdash \varphi$.

Proof. We proceed by induction on formula complexity. The atomic case, as well as the inductive steps for $\wedge$, $\vee$, and $\to$ are the same as in the completeness proof for IPC.

- Suppose $\varphi = \Box\psi$. Then, if $\varphi \in \Gamma$, by the definition of R^{C} and the forcing rule for $\Box$, $\mathcal{M}^{\mathsf{C}}, \Gamma \Vdash \Box\psi$.

  For the other direction, suppose $\varphi \notin \Gamma$. We want to find $\Gamma \preccurlyeq^{\mathsf{C}} \Gamma' \; R^{\mathsf{C}} \; \Delta$ such that $\psi \notin \Delta$. We construct Δ first. Note that $(\Gamma^{\Box}, \{\psi\})$ is a consistent pair, since otherwise $\Box\psi \in \Gamma$ by a standard argument. Take Δ to be a prime theory extending this pair. Next, we construct Γ'. Consider the pair

  $$(\Gamma \cup \{\Diamond\chi : \chi \in \Delta\}, \{\Box\theta : \theta \notin \Delta\}).$$

  This pair is consistent by the same argument that we used to check condition (FC1) in Proposition 3.6. Take Γ' to be a prime theory extending this pair. We have $\Gamma \subseteq \Gamma'$, so $\Gamma \preccurlyeq^{\mathsf{C}} \Gamma'$. Additionally, by construction $\Gamma'^{\Box} \subseteq \Delta \subseteq \Gamma'^{\Diamond}$, so $\Gamma' \; R^{\mathsf{C}} \; \Delta$.

- Suppose $\varphi = \Diamond\psi$. If $\varphi \notin \Gamma$, by the definition of R^{C} and the forcing rule for $\Diamond$, $\mathcal{M}^{\mathsf{C}}, \Gamma \nVdash \Diamond\psi$.

  If $\varphi \in \Gamma$, then we want to check that $(\Gamma^{\Box} \cup \{\psi\}, \mathcal{L} \setminus \Gamma^{\Diamond})$ is a consistent pair. If not, then for some $\theta \in \Gamma^{\Box}$ and $\chi \in \mathcal{L} \setminus \Gamma^{\Diamond}$ we would have $\vdash \theta \wedge \psi \to \chi$.[3] `Reg` then affords us $\vdash \Diamond(\theta \wedge \psi) \to \Diamond\chi$. As $\Box\theta \in \Gamma$, Lemma 1.1 implies $\Diamond(\theta \wedge \psi) \in \Gamma$. By deductive closure, $\Diamond\chi \in \Gamma$, which is a contradiction.

□

As usual, verifying that our canonical model satisfies the truth lemma immediately grants us completeness.

Theorem 3.8 IVL *is strongly complete with respect to* $\mathcal{V}$.

3.2 An Application of Completeness

Using soundness, one can quickly establish that for any φ, we have $\nvdash \nabla\varphi$; simply note that any one-point model can never force a borderline statement, as the borderlineness of φ requires at least two points, one forcing φ and one not. This raises the opposite question of when the system *settles the matter* of φ. There seem to be three natural candidates for how this should be formalized: $\vdash \varphi$ or $\vdash \neg\varphi$; $\vdash \neg\nabla\varphi$; and $\vdash \varphi \vee \neg\varphi$. In fact, these are all equivalent.

In order to prove this equivalence, we first establish two procedures for producing new models. The first is the construction of the *omnispective expansion.*

[3] We can take single formulas here since $\Gamma^{\Box}$ is closed under conjunction and $\mathcal{L} \setminus \Gamma^{\Diamond}$ is closed under disjunction.

Definition 3.9 *Let $\mathcal{M} = (W, \preccurlyeq, R, v)$ be an IV model and let o be some point not in W. Then the* omnispective expansion *of $\mathcal{M}$ by o is defined as $\mathcal{M}^o = (W \cup \{o\}, \preccurlyeq \cup \{(o, o)\}, R \cup (\{o\} \times W), v)$.*

The omnispective expansion of a model is then just the result of appending a new point that can access all of the pre-existing points via the modal relation but has no intuitionistic interaction with them.

Lemma 3.10 *Let $\mathcal{M} = (W, \preccurlyeq, R, v)$ be a vagueness model. Then $\mathcal{M}^o$ is a vagueness model. Moreover, if there are points $w_1, w_2 \in W$ with $\mathcal{M}, w_1 \Vdash \varphi$ and $\mathcal{M}, w_2 \nVdash \neg\varphi$, then $o \Vdash \nabla\varphi$.*

Corollary 3.11 *If there exists a model $\mathcal{M}$ with a point forcing φ and a point not forcing φ, then $\nvdash \neg\nabla\varphi$.*

Proof. $\mathcal{M}^o, o \Vdash \nabla\varphi$, so we are done by soundness. □

Corollary 3.12 *Let $\varphi \in \mathcal{L}$. Then at least one of φ and $\neg\varphi$ must be consistent with every set of the form $D(\Psi)$ where Ψ is a set of formulas whose negations are not theorems.*

Proof. Toward a contradiction, assume that φ is inconsistent with $D(\Psi_1)$ and $\neg\varphi$ is inconsistent with $D(\Psi_2)$ for some $\Psi_1, \Psi_2 \subseteq \mathcal{L}$ whose individual formulas are not refuted. By completeness, take a collection of models $\mathcal{M}_{i \in I}$ such that for each $\psi \in \Psi_1 \cup \Psi_2$, there is some i such that some point in $\mathcal{M}_i$ forces ψ. Now consider the omnispective expansion $(\bigsqcup_{i \in I} \mathcal{M}_i)^o$. The point o must force either φ or $\neg\varphi$ and also forces every formula in $D(\Psi_1)$ and $D(\Psi_2)$, which is a contradiction. □

We are now situated for the promised application of the completeness theorem.

Theorem 3.13 *The following are equivalent: (1) $\vdash \varphi$ or $\vdash \neg\varphi$; (2) $\vdash \neg\nabla\varphi$; and (3) $\vdash \varphi \vee \neg\varphi$.*

Proof. First we check (1) $\Longrightarrow$ (2). Suppose $\vdash \varphi$ or $\vdash \neg\varphi$. Then we have either $\vdash \Box\varphi$ or $\vdash \Box\neg\varphi$. In the second case, we additionally get $\vdash \neg\Diamond\varphi$. Both cases are then plainly inconsistent with $\nabla\varphi$, which is just shorthand for $\Diamond\varphi \wedge \neg\Box\varphi$.

For (2) $\Longrightarrow$ (3), we proceed by contraposition. If $\nvdash \varphi \vee \neg\varphi$, then there is a model $\mathcal{M}$ with a point x not forcing $\varphi \vee \neg\varphi$. Since $\neg\varphi$ is not forced, there is also some $x' \succcurlyeq x$ with $x' \Vdash \varphi$. Then we are done by Corollary 3.11.

For (3) $\Longrightarrow$ (1), we again use contraposition. By Corollary 3.12, only one of φ and $\neg\varphi$ can be inconsistent with a set of formulas of the form $D(\Psi)$. Assume that $\neg\varphi$ is consistent with all such sets. In the case that we have to choose φ for this role, the argument will be identical.

Take some maximally consistent set Γ containing φ. $D(\Gamma)$ is then consistent with $\neg\varphi$, so by strong completeness we get a model $\mathcal{N}'$ with a point y forcing $D(\Gamma) \cup \{\neg\varphi\}$. We also get a model $\mathcal{M}'$ forcing Γ at some point x. Note that we can take x and y to be maximal with respect to the intuitionistic relation since every point is underneath a maximal point in the canonical model. For ease of notation set $\mathcal{M} = \mathcal{M}'(x)$ and $\mathcal{N} = \mathcal{N}'(y)$. Then, writing $\mathcal{M} = (W_\mathcal{M}, \preccurlyeq_\mathcal{M}, R_\mathcal{M}, v_\mathcal{M})$ and $\mathcal{N} = (W_\mathcal{N}, \preccurlyeq_\mathcal{N}, R_\mathcal{N}, v_\mathcal{N})$, we define a new model $\mathcal{O} = (W, \preccurlyeq, R, v)$

as follows: $W = W_{\mathcal{N}} \sqcup W_{\mathcal{N}} \sqcup \{w\}$; $w \preccurlyeq w$, $w \preccurlyeq x$, and $w \preccurlyeq y$; for any $a \in W_{\mathcal{M}}$, $w\ R\ a$; for any $a \in W_{\mathcal{M}}$ and $z\ R_{\mathcal{N}}\ y$, $z\ R\ a$; the restriction of $\preccurlyeq$ to $\mathcal{M}$ is $\preccurlyeq_{\mathcal{M}}$ and the restriction of R to $\mathcal{M}$ is $R_{\mathcal{M}}$; the restriction of $\preccurlyeq$ to $\mathcal{N}$ is $\preccurlyeq_{\mathcal{N}}$, and the restriction of R to $\mathcal{N}$ is $R_{\mathcal{N}}$; no other instances of relations occur; and $v(p) = v_{\mathcal{M}}(p) \cup v_{\mathcal{N}}(p)$ for all $p \in \mathbf{P}$.

This construction is perhaps easiest understood pictorially, as presented in Figure 4. As an intuition pump, we want to mimic the standard argument of the disjunction property for IPC, so we add a new intuitionistic predecessor for x and y. Condition (FC3) forces us to allow w to be able to take modal steps to all points in $W_{\mathcal{M}}$. Then, (FC2) forces us to allow y to take modal steps to all points in $W_{\mathcal{M}}$. Finally, transitivity forces us to allow all z with $z\ R_{\mathcal{N}}\ y$ to be able to take modal steps to all points in $W_{\mathcal{M}}$.

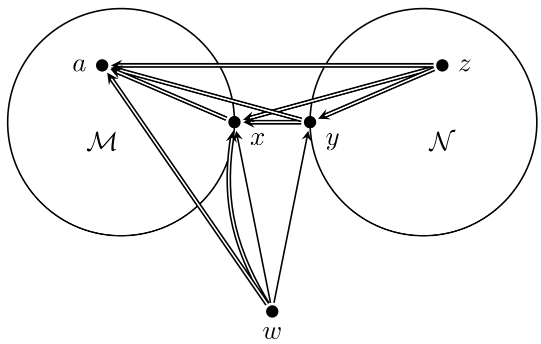


Fig. 4. The model $\mathcal{O}$ where some arbitrary $a \in W_{\mathcal{M}}$ and arbitrary z with $z\ R_{\mathcal{N}}\ y$ are shown

First we need to check that $\mathcal{O}$ is an IV model. This is almost entirely routine, with the exception that we are worried about satisfying (FC2) in the case where we have $z' \succcurlyeq z\ R\ a$. If this is the case, we must also have $z' \succcurlyeq_{\mathcal{N}} z\ R_{\mathcal{N}}\ y$. Then, by (FC2) for $\mathcal{N}$, there is some $y' \succcurlyeq_{\mathcal{N}} y$ such that $z'\ R_{\mathcal{N}}\ y'$. Since y is maximal, $z'\ R_{\mathcal{N}}\ y$. This means that $z'\ R\ a$, so a itself is a witness for this instance of (FC2).

It is clear that the forcing relation on $W_{\mathcal{M}(x)}$ is preserved, as no outgoing arrows were added for those points. We will prove by induction on the complexity of φ that for all $z \in W_{\mathcal{N}(y)}$, $\mathcal{N}, z \Vdash \varphi$ if and only if $\mathcal{O}, z \Vdash \varphi$.

- The atomic case is trivial by the construction of $\mathcal{O}$.
- The disjunction, conjunction, and conditional cases are immediate from the induction hypothesis, noting for the conditional case that there are no new outgoing instances of intuitionistic relations for points in $W_{\mathcal{N}}$.
- Suppose $\mathcal{N}, z \Vdash \Diamond\psi$. Then there is some $b \in \mathcal{N}$ with $\mathcal{N}, b \Vdash \psi$ and $z\ R\ b$. Such a b will still force ψ in $\mathcal{O}$ by the induction hypothesis, so $\mathcal{O}, z \Vdash \Diamond\psi$. In the other direction, suppose $\mathcal{O}, z \Vdash \Diamond\psi$. If this is witnessed in $W_{\mathcal{N}}$, we are done by the induction hypothesis, so suppose that $z\ R\ a$ for some $a \in \mathcal{M}$

with $\mathcal{O}, a \Vdash \psi$. We have $x \; R \; a$, so $\mathcal{M}, x \Vdash \Diamond\psi$, and $\Diamond\psi \in \Gamma$. Therefore, $\Diamond\Diamond\psi \in D(\Gamma)$, so $\mathcal{N}, y \Vdash \Diamond\psi$ by **4**$\Diamond$ and soundness. Now, since $z \; R \; a$, we can conclude that $z \; R_{\mathcal{N}} \; y$, as well, so $\mathcal{N}, z \vDash \Diamond\Diamond\psi$, implying our desired conclusion $\mathcal{N}, z \vDash \Diamond\psi$.

- Suppose $\mathcal{O}, z \Vdash \Box\psi$. We immediately get that $\mathcal{N}, z \Vdash \Box\psi$, by the induction hypothesis. In the other direction, we assume $\mathcal{N}, z \Vdash \Box\psi$. We need to make sure that there is no point $z' \succcurlyeq z$ with $z \; R \; a$ for some $a \in W_{\mathcal{M}}$ with $\mathcal{O}, a \nVdash \psi$. Suppose that this does happen. Then, by the maximality of x, $\mathcal{O}, x \Vdash \neg\Box\psi$. As we observed in the previous case of the inductive argument, $z' \; R \; a$ as well, so $\mathcal{O}, z' \Vdash \Diamond\neg\Box\psi$. This is inconsistent with $\mathcal{O}, z' \Vdash \Box\Box\psi$, which we have by **4**$\Box$ and soundness, so we arrive at a contradiction.

□

That the three most natural notions of φ being settled coincide in IVL seems to attest to the naturalness of the logic itself.

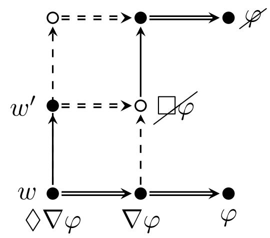
4 Higher-Order Vagueness

4.1 Stable Columnarity

Mormann [7] observes, in the classical setting, that when a logic at least as strong as S4 is assumed, all propositions are *stably columnar* [7]. That is, for any formula φ, $\nabla\nabla\varphi$ is provably equivalent to $\nabla\nabla\nabla\varphi$. By topological completeness for S4, this claim is just a redressing of the well-known fact that $\partial\partial A = \partial\partial\partial A$ where A is any subset of a topological space X and ∂ is the boundary (in the sense of closure minus interior) operator on X. This feature is sufficient for side-stepping paradoxes of higher-order vagueness, and it has the added thrust of not requiring the denial of clearly borderline cases. We can now verify that IVL proves that all formulas are stably columnar.

Lemma 4.1 $\vdash \Diamond\nabla p \to \nabla p$.

Proof. We can verify this quickly by drawing a picture and appealing to completeness.



As illustrated above, for any w forcing $\Diamond\nabla\varphi$, it must force $\Diamond\varphi$ by transitivity. Also, by (FC1), (FC2), and transitivity, any $w' \succcurlyeq w$ cannot force $\Box\varphi$. Therefore, $\vdash \Diamond\nabla\varphi \to \nabla\varphi$. □

The above lemma then allows us derive a more useful theorem of the system.

Proposition 4.2 $\vdash \neg\Box\nabla\nabla p$.

Proof.

$\vdash \Diamond\nabla p \to \nabla p$	Lemma 4.1	(1)
$\vdash \Box\Diamond\nabla p \to \Box\nabla p$	$\mathtt{Reg}$, (1)	(2)
$\vdash \Box\nabla p \to \Diamond\Box\nabla p$	$\mathbf{T}\Diamond$	(3)
$\vdash \Box\Diamond\nabla p \to \Diamond\Box\nabla p$	(2), (3)	(4)
$\vdash \Box\nabla\nabla p \to \Box\Diamond\nabla p$	$\mathbf{K}\Box\mathbf{a}$	(5)
$\vdash \Box\nabla\nabla p \to \Diamond\Box\nabla p$	(4), (5)	(6)
$\vdash \Box\nabla\nabla p \to \Box\neg\Box\nabla p$	$\mathbf{K}\Box\mathbf{a}$	(7)
$\vdash \Box\neg\Box\nabla p \to \neg\Diamond\Box\nabla p$	Proposition 1.2	(8)
$\vdash \Box\nabla\nabla p \to \neg\Diamond\Box\nabla p$	(7), (8)	(9)
$\vdash \neg\Box\nabla\nabla p$	(6), (9)	(10)

□

At this point we need only put together some established facts to prove the stable columnarity theorem for IVL.

Theorem 4.3 $\vdash \nabla\nabla p \leftrightarrow \nabla\nabla\nabla p$.

Proof. The theorem is of course equivalent to the pair of claims $\vdash \nabla\nabla\nabla p \to \nabla\nabla p$ and $\vdash \nabla\nabla p \to \nabla\nabla\nabla p$. To verify the first claim, we simply note that $\nabla\nabla\nabla p \to \Diamond\nabla\nabla p$ and $\Diamond\nabla\nabla p \to \nabla\nabla p$ are both theorems, the first by conjunction elimination and the second by Lemma 4.1. For the second claim, since $\nabla\nabla\nabla p$ is shorthand for $\Diamond\nabla\nabla p \wedge \neg\Box\nabla\nabla p$, we just need to verify that $\nabla\nabla p \to \Diamond\nabla\nabla p$ and $\nabla\nabla p \to \neg\Box\nabla\nabla p$ are theorems. As $\nabla\nabla p \to \Diamond\nabla\nabla p$ is just an instance of $\mathbf{T}\Diamond$ and $\vdash \nabla\nabla p \to \neg\Box\nabla\nabla p$ holds trivially in light of Proposition 4.2, we are done. □

4.2 Axiom M

Bobzien and Rumfitt are also concerned with paradoxes of higher-order vagueness. Partially in an effort to block such problems, they consider and defend the axiom $\neg\Box\nabla p$ (**M**). (Note that this is classically equivalent to the McKinsey axiom $\Box\Diamond p \to \Diamond\Box p$.) Part of their charge for an adequate intuitionistic modal logic of vagueness is that it validates the equivalence of $\nabla p \leftrightarrow \nabla\nabla p$ (the $\nabla\nabla$ principle) and **M**. Therefore, by endorsing **M**, they deny that there is a "real hierarchy" of higher-order vagueness, a move that is in keeping with Wright's position [3, p. 244][9]. Since Bobzien and Wright prove that their axioms and rules are strong enough to derive this equivalence, we can be certain that IVL is sufficiently strong, as well, as it is a strengthening. We will show that our methods can accommodate the intuitionist who assents to **M**.

We define the logic $\mathsf{IVLM} = \mathsf{IVL} \oplus \{\mathbf{M}\}$. For the sake cleaner notation, $\vdash_{\mathsf{M}}$ will denote provability in IVLM. Semantics for IVLM are easy to come by, since we can simply keep the forcing clauses from our semantics for IVL and restrict to a smaller class of frames. First we define a new kind of point that can occur in S4 Fischer Servi frames.

Definition 4.4 *Let* $(W, \preccurlyeq, R)$ *be an* S4 *Fischer Servi frame. We say that* $x \in W$ *is quasi-isolated if for all* y *with* $x\ R\ y$ *there exists an* $x'' \succcurlyeq x$ *such that for all* $x''' \succcurlyeq x''$ *and for all* z *with* $x'''\ R\ z$*, we have* $y \preccurlyeq z$*.*

As the name suggests, quasi-isolated points are meant to be a birelational stand-in for isolated points in a topological space or maximal points in a unirelational structure. In the classical setting, such points are the ones that validate $\Diamond\varphi \to \Box\varphi$. This is classically equivalent to $\neg\nabla\varphi$, so the hope is that in our setting, quasi-isolated points similarly can resist making borderline statements true. The following proposition illustrates this.

Proposition 4.5 *Let* $\mathcal{M} = (W, \preccurlyeq, R, v)$ *be an* S4 *Fischer-Servi model. Then, if* x *is quasi-isolated,* $x \nVdash \nabla\varphi$ *for any formula* φ*.*

Proof. We may assume that $x \Vdash \Diamond\varphi$. If this is the case, there is y with $x\ R\ y$ such that $y \Vdash \varphi$. Because x is quasi-isolated, we can find a point $x'' \succcurlyeq x$ such that every modal successor of an intuitionistic successor of x'' is additionally an intuitionistic successor of y. By persistence, we must have that $x'' \Vdash \Box\varphi$. This means that $x \nVdash \neg\Box\varphi$, so $x \nVdash \nabla\varphi$. □

Another useful fact is that over the class $\mathcal{V}$, there is a simpler equivalent condition to quasi-isolatedness.

Proposition 4.6 *Let* $(W, \preccurlyeq, R)$ *be an IV frame. Then, a point* x *is quasi-isolated if and only if for every* y *with* $x\ R\ y$*, there is an* $x' \succcurlyeq x$ *such that for all* z *with* $x'\ R\ z$*, we have* $y \preccurlyeq z$*.*

Proof. The left-to-right direction is trivial, as the new condition is evidently weaker. In the other direction, suppose that x satisfies the weaker condition and that $x\ R\ y$. We get an $x' \succcurlyeq x$ such that all of its modal successors are intuitionistic successors of y. Because we are working in an IV frame, x' has a diamond-reflection point x''. Now, for any x''' and z with $x'' \preccurlyeq x'''\ R\ z$, we have $x'\ R\ z$. But by assumption, this means that $y \preccurlyeq z$, so x is quasi-isolated. □

We are now in a position to write down our new frame condition:

(FC4-weak) For every $w \in W$ there exist w' and x with $w \preccurlyeq w'\ R\ x$ such that x is quasi-isolated.

We leave open whether this frame condition actually corresponds to **M**. Analogous to the situation with (FC3-weak) and (FC3), we can actually strengthen this condition without losing completeness.

(FC4) For every $w \in W$ there exists x with $w\ R\ x$ such that x is quasi-isolated.

This condition is illustrated in Figure 5. An IV frame satisfying (FC4) is called a *weakly quasi-scattered intuitionistic vagueness* (WQSIV) frame. We call the class of such frames $\mathcal{W}$.

Theorem 4.7 IVLM *is sound with respect to* $\mathcal{W}$*.*

Proof. This follows from Theorem 2.6 and Propositions 4.5 and 4.6. □

$\forall w \exists x \forall y \exists x'$

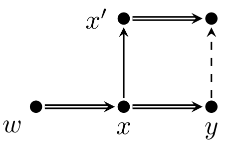


Fig. 5. The diagram for condition (FC4)

As with IVL, we use soundness to prove that IVLM is conservative over IPC.

Corollary 4.8 IVLM *is a conservative extension of* IPC.

Proof. Let $\varphi \in \mathcal{L}_0$ by a non-theorem of IPC. Then there is a some model $\mathcal{M} = (W, \preccurlyeq, v)$ with $\mathcal{M}, w \nVdash \varphi$ for some $w \in W$. For some $c \notin W$, we define $\mathcal{M}^\dagger = (W \cup \{c\}, \preccurlyeq \cup \{c, c\}, W^2 \cup (W \times \{c\}), v)$. One can check that $\mathcal{M}^\dagger$ is a WQSIV model, and since the intuitionistic relationship was left alone, we have $\mathcal{M}^\dagger, w \nVdash \varphi$. By soundness, φ is not a theorem of IVLM. □

4.3 Ersatz Topology

Before we can set out to prove a completeness theorem for IVLM and $\mathcal{W}$, we will need to prove a handful of topologically-inspired facts. As we will see, the system IVL is strong enough to force our modalities $\Box$, $\Diamond$, and ∇ to behave sufficiently similarly to the topological operators interior, closure, and boundary, respectively. In this subsection, as well as in §4.3, we will use completeness of IVL for $\mathcal{V}$ to avoid particularly arduous derivations. Of course, none of these applications are strictly speaking necessary. In all of the following results, $\mathsf{L} = \mathsf{IVL} \oplus \Gamma$, where Γ is some set of formulas.

Definition 4.9 *A formula φ is* L-nowhere dense *if* $\vdash_{\mathsf{L}} \neg\Box\Diamond\varphi$.

Lemma 4.10 *If φ is* L*-nowhere dense, then* $\vdash_{\mathsf{L}} \varphi \to \nabla\varphi$.

Proof. By $\mathbf{T}\Diamond$, $\vdash_{\mathsf{L}} \varphi \to \Diamond\varphi$. Then by `Reg`, we see $\vdash_{\mathsf{L}} \Box\varphi \to \Box\Diamond\varphi$. Contraposition yields $\vdash_{\mathsf{L}} \neg\Box\Diamond\varphi \to \neg\Box\varphi$, so by `MP`, $\vdash \neg\Box\varphi$. We can conclude $\vdash_{\mathsf{L}} \varphi \to \nabla\varphi$. □

Lemma 4.11 L*-nowhere denseness is preserved by disjunction.*

Proof. Let φ and ψ be L-nowhere dense formulas. By necessitation, we have both $\vdash_{\mathsf{L}} \Box\neg\Box\Diamond\varphi$ and $\vdash_{\mathsf{L}} \Box\neg\Box\Diamond\psi$. We will just need to prove $\vdash ((\Box\neg\Box\Diamond\varphi) \wedge (\Box\neg\Box\Diamond\psi)) \to \neg\Box\Diamond(\varphi \vee \psi)$. Note the use of $\vdash$. Regardless of which logic L is, we will prove this implication in the weaker system IVL so that we can avail ourselves of the completeness theorem that we already proved. We proceed by contradiction. Then $\Box\neg\Box\Diamond\varphi$, $\Box\neg\Box\Diamond\psi$, and $\Box\Diamond(\varphi\vee\psi)$ are mutually consistent, so we can find a model $\mathcal{M}$ with a point w forcing all three. Since w forces $\neg\Box\Diamond\varphi$, we can find $w' \succcurlyeq w$ and x with $w' \; R \; x$ such that $x \nVdash \Diamond\varphi$. Now, let x' be a diamond-reflection point of x. We must have $x' \Vdash \neg\Diamond\varphi$. As $x' \Vdash \neg\Box\Diamond\psi$, it has an intuitionistic successor x'' which in turn has a modal successor y, with $y \nVdash \Diamond\psi$. Again, we take a diamond-reflection point y' of y. Then $y' \Vdash \neg\Diamond\varphi \wedge \neg\Diamond\psi$, but this is equivalent to $\neg\Diamond(\varphi \vee \psi)$, contradicting that $w \Vdash \Box\Diamond(\varphi \vee \psi)$. Figure

6 provides a helpful visual reference for this argument. □

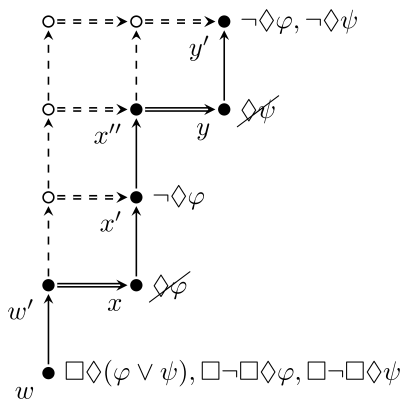


Fig. 6. For the proof of Lemma 4.11, a partial view of $\mathcal{M}$, where x' and y' are diamond-reflection points for x and y, respectively.

There is also a natural analog of topological closedness for a system L, which will furnish elegant proofs of facts about ∇.

Definition 4.12 *A formula* φ *is* L*-closed if* $\vdash_{\mathsf{L}} \Diamond\varphi \to \varphi$.

We can promptly make a couple of observations about L-closed formualas. First, by $\mathbf{T}\Diamond$, an L-closed formula φ is always provably equivalent to $\Diamond\varphi$. Second, any formula of the form $\Diamond\varphi$ is L-closed by $\mathbf{4}\Diamond$. Finally, by Lemma 4.1, all formulas of the form $\nabla\varphi$ are L-closed.

We can also prove a proposition about such formulas mirroring the topological fact that the boundary of a closed set A is always a subset of A.

Proposition 4.13 *If* φ *is* L*-closed, then* $\vdash_{\mathsf{L}} \nabla\varphi \to \varphi$.

Proof. Clearly we have $\vdash \nabla\varphi \to \Diamond\varphi$, so $\vdash \nabla\varphi \to \varphi$, as φ is L-closed. □

Corollary 4.14 $\vdash_{\mathsf{L}} \nabla\nabla p \to \nabla p$.

Proof. This follows immediately from Lemma 4.1 and Proposition 4.13. □

As a corollary, we obtain Bobzien and Rumfitt's $\nabla\nabla$ principle.

Corollary 4.15 $\vdash_{\mathsf{M}} \nabla p \leftrightarrow \nabla\nabla p$.

Proof. Since ∇p is IVLM-closed, $\vdash_{\mathsf{M}} \neg\Box\nabla p \to \neg\Box\Diamond\nabla p$. Therefore, $\vdash_{\mathsf{M}} \neg\Box\Diamond\nabla p$, so ∇p is IVLM-nowhere dense. The result then follows from Lemma 4.10 and Corollary 4.14. □

4.4 Another Completeness Theorem

We have now positioned ourselves to prove a completeness theorem for IVLM.

Theorem 4.16 IVLM *is strongly complete with respect to* $\mathcal{W}$.

Proof. We define the canonical model $\mathcal{M}^{\mathsf{CM}} = (W^{\mathsf{CM}}, \preccurlyeq^{\mathsf{CM}}, R^{\mathsf{CM}}, v^{\mathsf{CM}})$ for IVLM almost exactly as we defined $\mathcal{M}^{\mathsf{C}}$, with the only difference being that our prime theories are now defined in terms of $\vdash_{\mathsf{M}}$ instead of $\vdash$. All of the previous results still hold, but we need to check that the canonical model now also satisfies condition (FC4). Let Γ be a prime theory. First we claim that we need only find a prime theory Δ such that $\Gamma^{\Box} \subseteq \Delta \subseteq \Gamma^{\Diamond}$ and Δ has the property $\Diamond\varphi \in \Delta \Longrightarrow \neg\Box\varphi \notin \Delta$ for all $\varphi \in \mathcal{L}$.

Suppose that such a Δ exists, and suppose further that $\Delta\ R^{\mathsf{CM}}\ \Theta$. We need to check that there is some extension Δ' of Δ such that $\Delta'^{\Box} \supseteq \Theta$. This is exactly the condition guaranteeing that any modal successor of Δ' is an intuitionistic successor of Θ. By Lemma 3.4, this is equivalent to the consistency of $\Delta \cup B(\Theta)$. Toward a contradiction, suppose that $\Delta \cup B(\Theta)$ is inconsistent. Then $\Delta \vdash_{\mathsf{M}} \neg(\wedge_{i=1}^{k} \Box\varphi_i)$ for some set of formulas $\{\varphi_i\}_{i=1}^{k} \subseteq \Theta$. Using $\mathbf{K\Box a}$ and noting that Θ is closed under conjunction, we can rewrite this as $\Delta \vdash \neg\Box\varphi$ for some $\varphi \in \Theta$. Since $\Delta\ R^{\mathsf{CM}}\ \Theta$, however, $\Diamond\varphi \in \Delta$. Therefore, by the assumption on Δ, $\neg\Box\varphi \notin \Delta$, which is our desired contradiction.

Now we obtain the desired set Δ. The condition that we want Δ to satisfy is equivalent to Δ not containing any formulas of the form $\nabla\varphi$, so we consider the pair $(\Gamma^{\Box}, (\mathcal{L} \setminus \Gamma^{\Diamond}) \cup N(\mathcal{L}))$. If we can prove the consistency of this pair, we are again done by Lemma 3.4. Suppose that the pair is not consistent. Then there are formulas $\varphi \in \Gamma^{\Box}$ and $\psi \notin \Gamma^{\Diamond}$ and a set of formulas $\{\theta_i\}_{i=1}^{k} \subseteq N(\Theta)$ such that $\vdash \varphi \to \psi \vee (\vee_{i=1}^{k} \theta_i)$. Intuitionistic reasoning grants us $\vdash (\varphi \wedge \neg\psi) \to \vee_{i=1}^{k} \theta_i$. Applying `Reg` twice, $\vdash \Diamond\Box(\varphi \wedge \neg\psi) \to \Diamond\Box(\vee_{i=1}^{k} \nabla\theta_i)$, which is equivalent to $\vdash \Diamond(\Box\varphi \wedge \Box\neg\psi) \to \Diamond\Box(\vee_{i=1}^{k} \nabla\theta_i)$. We now check that $\Gamma \vdash \Diamond\Box\neg\psi$. If not, then by Lemma 2.9 and the disjunctivity of Γ, we would have $\Gamma \vdash \Diamond\neg\Box\neg\psi$, which is equivalent by Lemma 1.2 to $\Gamma \vdash \Diamond\neg\neg\Diamond\psi$. By $\mathbf{S}\Diamond$, this is equivalent to $\Gamma \vdash \Diamond\Diamond\psi$, which, using $\mathbf{4}\Diamond$, implies $\Gamma \vdash \Diamond\psi$, which is a contradiction. Therefore, $\Gamma \vdash \Diamond\Box\neg\psi$. By $\mathbf{4}\Box$, $\Gamma \vdash \Box\Box\varphi$, so by Lemma 1.1, $\Gamma \vdash \Diamond(\Box\varphi \wedge \Box\neg\psi)$. Therefore, $\Gamma \vdash \Diamond\Box(\vee_{i=1}^{k} \nabla\theta_i)$. Each $\nabla\theta_i$ is IVLM-nowhere dense, so by Proposition 4.11 the disjunction is as well. By Proposition 4.10, $\vdash_{\mathsf{M}} (\vee_{i=1}^{k} \nabla\theta_i) \to \nabla(\vee_{i=1}^{k} \nabla\theta_i)$. Again, applying `Reg` twice, we have $\vdash_{\mathsf{M}} \Diamond\Box(\vee_{i=1}^{k} \nabla\theta_i) \to \Diamond\Box\nabla(\vee_{i=1}^{k} \nabla\theta_i)$. Hence, $\Gamma \vdash_{\mathsf{M}} \Diamond\Box\nabla\theta$ for some θ. But $\mathbf{M}$ affords us $\vdash \Box\nabla\theta \leftrightarrow \bot$, so we have $\Gamma \vdash \Diamond\bot$, which contradicts $\mathbf{K\Diamond b}$. Γ is then inconsistent, which is a contradiction. $\Box$

Thus, we have shown that the semantics developed in Section 2 can be easily adapted for use by proponents of both columnar vagueness and an intuitionistic logic of vagueness without losing the mathematical power of completeness. In particular, this gives a strong formal grounding for the modal extension of Wright's view.

5 Conclusion

In this paper, we have furthered the work of Bobzien and Rumfitt to formalize and modalize Wright's intuitonsitic position on the Sorites paradox. Like they did, we propose a deductive system, but we also establish a semantics for which our system is sound and complete, thereby allowing us to establish results of philosophical import, like verifying that the underlying sentential calculus really is intuitionistic.

We have left open a few mathematical questions. Establishing the finite model property for IVL or IVLM and settling whether there are first-order correspondents for the axiom $\mathbf{M}$ over the class $\mathcal{S}$ both seem to be natural next steps. Additionally, while we do not think $\mathbf{S}\nabla$ is philosophically compelling in the absence of $\mathbf{S}\Diamond$, establishing whether there is a first-order definable class of frames for which we have a completeness theorem could be of technical interest.

In [2], Bobzien works with semantics for a predicate extension of S4.1 and manages to give an account of the frames in terms of viewpoints. Although our semantics have proven useful, we have not given any sort of intuitive gloss on them, so a natural extension of Bobzien's recent work would be to argue for the philosophical meaningfulness of the semantics. Additionally, performing a parallel analysis of a predicate extension of IVL or IVLM would bolster the intuitionist's formal foothold.

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