A Semantics for a Failed Axiomatization of K

Hitoshi Omori [1] Daniel Skurt [2]

Department of Philosophy I, Ruhr University Bochum
Universitätsstraße 150, 44780, Bochum, Germany

Abstract

In "Yet another "choice of primitives" warning: Normal modal logics", Lloyd Humberstone discussed a failed axiomatization for the normal modal logic K with $\Diamond$ as the only primitive modal operator. More specifically, Humberstone observed that a simple translation of the standard axiomatization for K, where all occurrences of the necessity operator $\Box$ are replaced by $\neg\Diamond\neg$, will not be a complete axiomatization, since $\Diamond p \to \Diamond\neg\neg p$ is not derivable. As a result, the emerging proof system resists the standard Kripke semantics. However, to the best of the authors' knowledge, no semantics for the failed axiomatization of K is known in the literature. The aim of this article is to offer the first sound and complete semantics for the failed axiomatization of K by making use of a semantical framework suggested by John Kearns. In short, Kearns' semantics is a combination of non-deterministic semantics together with an additional hierarchy of valuations. We will also discuss a small question left open by Humberstone in the same paper. In view of the results presented in this article, we hope to establish part of the versatility of Kearns' semantics.

Keywords: Non-deterministic Semantics, Primitive Connectives, Normal Modal Logics.

1 Introduction

Both in classical and nonclassical logics, there is a freedom in choosing the set of primitive connectives. For example, in classical logic, one may take negation and the conditional as primitive connectives, or take all, negation, conjunction, disjunction and conditional as primitive. Or even one single connective known as Sheffer's stroke.

[1] The work reported in this paper was initially supported by JSPS KAKENHI Grant Number JP18K12183, and later by a Sofja Kovalevskaja Award of the Alexander von Humboldt-Foundation, funded by the German Ministry for Education and Research. Some part of the results were presented at *Oberseminar Logik und Sprachtheorie* in Tübingen in May 2019. We would also like to thank the three referees for their careful reading and helpful comments, suggestions and corrections that improved the paper. Email: Hitoshi.Omori@rub.de

[2] Some of the observations of this article were presented at *Trends in Logic XVIII* in Milan in September 2018, at the *Kolloquium Philosophie & Linguistik* in Göttingen in November 2018, at the *Philosophisches Kolloquium* in Leipzig in December 2018. DS would like to thank the audience for their helpful comments and encouragements. Email: Daniel.Skurt@rub.de

We also know, however, that sometimes some additional care is required. [3] For example, if we take negation and disjunction as primitive connectives, then the following set of axioms and the rule of inference, due to Hilbert and Ackermann, are complete with respect to the usual two-valued semantics, where $A \to B$ abbreviates $\neg A \vee B$.

$$(A \vee A) \to A \qquad (A \vee B) \to (B \vee A)$$
$$A \to (A \vee B) \qquad (A \to B) \to ((C \vee A) \to (C \vee B))$$
From A and $A \to B$, infer B

Now, consider negation and conjunction as primitive connectives, and if we simply translate the above set of axioms and the rule of inference with the usual definitions $A \vee B =_{\text{def.}} \neg(\neg A \wedge \neg B)$ and $A \to B =_{\text{def.}} \neg(A \wedge \neg B)$, then we obtain the following:

$$\neg((\neg(\neg A \wedge \neg A)) \wedge \neg A) \qquad \neg(A \wedge \neg\neg(\neg A \wedge \neg B)$$
$$\neg(\neg(\neg A \wedge \neg B) \wedge \neg\neg(\neg B \wedge \neg A))$$
$$\neg((\neg(A \wedge \neg B)) \wedge \neg\neg(\neg(\neg C \wedge \neg A) \wedge \neg\neg(\neg C \wedge \neg B)))$$
From A and $\neg(A \wedge \neg B)$, infer B

However, as observed by Henryk Hiż in [8], the latter system is *not* a complete axiomatization since we may observe that $\neg(\neg p \wedge p)$ is not derivable.

As for modal logics, it was shown by David Makinson in [14] that "*the decision whether to treat the zero-ary falsum operator as primitive or as defined, affects the general structure of the lattice of all modal logics.*" Moreover, in [9], Lloyd Humberstone observed, among other things, that a simple translation of the axiomatization for the modal logic K with the necessity (or "box") as the primitive connective, obtained by replacing the occurrences of "box" by "not diamond not" will *not* be a complete axiomatization, since we may observe that $\Diamond p \to \Diamond\neg\neg p$ is *not* derivable. [4]

Note, however, that a sound and complete semantics for the failed axiomatization of K, which we refer to as $\mathsf{K_f}$, is not yet available in the literature, at least to the best of the authors' knowledge. [5]

Based on these, the aim of this article is to fill in this gap and as a byproduct show the versatility of John Kearns' semantics, devised in [12]. [6] More specifically, we will first introduce a sixteen-valued non-deterministic semantics (cf. [1] for a survey) with an additional hierarchy on the set of all valuations for

---

[3] For an interesting discussion related to this point, but from a wider perspective, see [7] and references therein.

[4] This is also reported by Richmond Thomason in a recent note [21] without any reference to Humberstone's observation. We will not discuss Thomason's note since the eight-valued matrix he introduces seems to be not fully articulated. Note that, as pointed out by Humberstone, there is a four-valued matrix that will establish one of Thomason's results (see Remark 2.6 below).

[5] This is not to say that there are no sound and complete Kripke semantics for the modal logic K with a primitive possibility operator, see for example [2]. In brief, this axiomatization makes use of one more axiom than just the K-axiom. We will return to this point later.

[6] Kearns' semantics was later applied to a larger family of normal modal logics in [16].

$\mathsf{K_f}$ following Kearns. Then we will prove that $\mathsf{K_f}$ is sound and complete with respect to our Kearns' style semantics, now involving sixteen values, instead of four values. [7] Once these are established, we will extend $\mathsf{K_f}$ with additional axioms in order to give a sound and complete semantics for a system we call $\mathsf{S5_f}$. With this semantics we will deal with a problem left open by Humberstone, namely showing the independence of $\Diamond\neg\neg p \to \Diamond p$ from the failed axiomatization.

2 Semantics and proof system

Our language $\mathcal{L}$ consists of the set $\{\neg, \Diamond, \to\}$ of propositional connectives and a countable set Prop of propositional parameters. Furthermore, we denote by Form the set of formulas defined as usual in $\mathcal{L}$. We denote formulas of $\mathcal{L}$ by A, B, C, etc. and sets of formulas of $\mathcal{L}$ by Γ, Δ, Σ, etc.

2.1 Proof system

We first introduce the target system of this article, namely the system $\mathsf{K_f}$. We also define a subsystem that will be sound and complete with respect to the non-deterministic semantics *without* the hierarchy.

Definition 2.1 First, the system $\mathsf{K_f}$ consists of the following axiom schemata and rules of inference: [8]

$$A \to (B \to A) \quad \text{(Ax1)} \qquad (A\to(B\to C))\to((A\to B)\to(A\to C)) \quad \text{(Ax2)}$$

$$(\neg B\to\neg A)\to(A\to B) \quad \text{(Ax3)} \qquad \neg\Diamond\neg(A\to B)\to(\neg\Diamond\neg A\to\neg\Diamond\neg B) \quad \text{(LK)}$$

$$\frac{A \quad A\to B}{B} \quad \text{(MP)} \qquad \frac{A}{\neg\Diamond\neg A} \quad \text{(RN)}$$

We write $\vdash_{\mathsf{K_f}} A$ for, there is a *proof for* A in K_f if there is a sequence of formulas $B_1, \dots, B_n, A$ $(n \geq 0)$, such that every formula in the sequence either (i) is an axiom of $\mathsf{K_f}$; or (ii) is obtained by (MP) or (RN) from formulas preceding it in the sequence. Moreover, we define $\Gamma \vdash_{\mathsf{K_f}} A$ iff for a finite subset Γ' of Γ, $\vdash_{\mathsf{K_f}} C_1 \to (C_2 \to (\cdots (C_n \to A) \cdots))$ where $C_i \in \Gamma' (1 \leq i \leq n)$.

Second, we define a subsystem of $\mathsf{K_f}$, referred to as $\mathsf{k_f}$, which is obtained by eliminating (RN) and adding the following schemata: [9]

$$\Diamond\neg\neg(A\to B)\to(\neg\Diamond\neg A\to\Diamond\neg\neg B) \quad (\mathrm{Ak_f1}) \qquad \neg\Diamond\neg\neg(A\to B)\to\neg\Diamond\neg A \quad (\mathrm{Ak_f2})$$

$$\neg\Diamond\neg\neg(A\to B)\to\neg\Diamond\neg\neg B \quad (\mathrm{Ak_f3}) \qquad \Diamond\neg(A\to B)\to\Diamond\neg B \quad (\mathrm{Ak_f4})$$

$$\Diamond\neg\neg\neg A\to\Diamond\neg A \quad (\mathrm{Ak_f5}) \qquad \Diamond\neg A\to\Diamond\neg\neg\neg A \quad (\mathrm{Ak_f6})$$

We define $\Gamma \vdash_{\mathsf{k_f}} A$ (A *can be derived from* Γ) iff there is a sequence of formulas $B_1, \dots, B_n, A$ $(n \geq 0)$, such that every formula in the sequence either (i) is an

[7] Sixteen values may remind us of the system $\mathrm{SIXTEEN}_3$ of Yaroslav Shramko and Heinrich Wansing (cf. [18,19]). However, we could not establish any relation between their semantics and our semantics.

[8] Where (Ax1), (Ax2), (Ax3) and (MP) are a well-known axiomatization of classical propositional logic (cf. [20]) and, (LK) and (RN) are the K-axiom and rule of necessitation expressed in terms of $\neg$ and $\Diamond$.

[9] We would like to thank one of the anonymous reviewers for pointing out the missing axioms.

element of Γ (ii) is an axiom of $\mathsf{k_f}$; or (iii) is obtained by (MP) from formulas preceding it in the sequence.

Remark 2.2 It is rather easy to see that $\mathsf{k_f}$ is a subsystem of $\mathsf{K_f}$. Indeed, note first that in $\mathsf{K_f}$, we have the following rules in view of (LK), (MP) and (RN):

$$\frac{A\to B}{\neg\Diamond\neg A\to\neg\Diamond\neg B}\ ,\quad \frac{A\to(B\to C)}{\neg\Diamond\neg A\to(\neg\Diamond\neg B\to\neg\Diamond\neg C)}.$$

Then, in order to see that (Ak$_\mathsf{f}$1) is derivable in $\mathsf{K_f}$, apply the second rule to $A\to(\neg B\to\neg(A\to B))$. For the rest, apply the first rule to $\neg(A\to B)\to A$, $\neg(A\to B)\to\neg B$, $B\to(A\to B)$, $A\to\neg\neg A$ and $\neg\neg A\to A$, respectively.

2.2 A detour: counter-model of Humberstone

Here, we will review the counter-model used by Humberstone, in [9], to establish that $\Diamond A\to\Diamond\neg\neg A$ is *not* derivable in $\mathsf{K_f}$.

Definition 2.3 [Humberstone] A model for $\mathcal{L}$ is a triple $\langle W, N, V\rangle$ in which W is a set with $\emptyset\neq N\subseteq W$ and V is a function assigning to each propositional variable a subset of W. Given a model $\mathcal{M}=\langle W,N,V\rangle$ we define truth of a formula A at a point $u\in W$ ($\mathcal{M}\models_u A$) as follows:

- $\mathcal{M}\models_u p$ iff $u\in V(p)$, if $p\in\mathsf{Prop}$;
- $\mathcal{M}\models_u B\to C$ iff $\mathcal{M}\not\models_u B$ or $\mathcal{M}\models_u C$;
- $\mathcal{M}\models_u \neg B$ iff $u\in N$ and $\mathcal{M}\not\models_u B$;
- $\mathcal{M}\models_u \Diamond B$ iff for some $v\in W:\mathcal{M}\models_v B$.

A formula A is true in the model $\mathcal{M}=\langle W,N,V\rangle$, (notation: $\mathcal{M}\models A$), just in case for all $u\in N$, we have $\mathcal{M}\models_u A$, and valid (notation: $\models_H A$, where H stands for Humberstone) if it is true in every model.

Fact 2.4 (Theorem 2.1 in [9]) *For all $A\in\mathsf{Form}$, if $\vdash_{\mathsf{K_f}} A$ then $\models_H A$.*

Fact 2.5 (Corollary 2.2 in [9]) $\not\models_H \Diamond p\to\Diamond\neg\neg p$.[10]

Proof. Consider a two-element model $\mathcal{M}_0=\langle W_0,N_0,V_0\rangle$, with $W_0=\{u,v\}$ and $u\neq v$, $N_0=\{u\}$ and $V_0(p)=\{v\}$. Now, we have $\mathcal{M}_0\models_u\Diamond p$, but $\mathcal{M}_0\not\models_u\Diamond\neg\neg p$. The latter follows since there is no element in W_0, such that $\neg\neg p$ is true. Indeed, for u, we have $u\in N$ but $\mathcal{M}_0\not\models_u p$, and for v, we have $\mathcal{M}_0\not\models_v\neg p$, but $v\notin N$. Therefore, $\mathcal{M}_0\not\models_u\Diamond p\to\Diamond\neg\neg p$. as desired. $\Box$

Remark 2.6 Note that the above model $\mathcal{M}_0$ can be seen as a four-valued matrix with its four elements being $1=\{u,v\}, 2=\{u\}, 3=\{v\}$ and $4=\emptyset$, and designated values 1 and 2. Truth tables for the connectives are as follows.

A	$\neg A$	$\Diamond A$
1	4	1
2	4	1
3	2	1
4	2	4

$A\to B$	1	2	3	4
1	1	2	3	4
2	1	1	3	3
3	1	2	1	2
4	1	1	1	1

[10] Thus, $\mathsf{K_f}$ does not enjoy the replacement property, also known as self-extensionality. So, if this property is crucial for modal logics (cf. [15]), then $\mathsf{K_f}$ is not a modal logic.

Then, if we assign the value 3 to p, then $\Diamond p \to \Diamond\neg\neg p$ receives the value 4, as desired. Note, however, that $\Diamond\neg\neg p \to \Diamond p$ will be verified in this model (note that the above matrix can be found in [9, Figure 1]).

2.3 Semantics

We now turn to present the semantics for $\mathsf{K_f}$. To this end, we first introduce the basic Nmatrix which requires sixteen truth values.

Definition 2.7 A $\mathsf{K_f}$-*Nmatrix* for $\mathcal{L}$ is a tuple $M = \langle \mathcal{V}, \mathcal{T}, \mathcal{O}\rangle$, where:

(a) $\mathcal{V} = \{\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}, \mathbf{T_4}, \mathbf{t_1}, \mathbf{t_2}, \mathbf{t_3}, \mathbf{t_4}, \mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}, \mathbf{f_4}, \mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}, \mathbf{F_4}\}$,

(b) $\mathcal{T} = \{\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}, \mathbf{T_4}, \mathbf{t_1}, \mathbf{t_2}, \mathbf{t_3}, \mathbf{t_4}\}$,

(c) For every n-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{*}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} \setminus \{\emptyset\}$ as follows (we omit the brackets for sets):

A	$\tilde{\neg}A$	$\Diamond A$	A	$\tilde{\neg}A$	$\Diamond A$	A	$\tilde{\neg}A$	$\Diamond A$	A	$\tilde{\neg}A$	$\Diamond A$
$\mathbf{T_1}$	$\mathbf{F_4}$	$\mathcal{T}$	$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathcal{T}$	$\mathbf{f_1}$	$\mathbf{t_4}$	$\mathcal{T}$	$\mathbf{F_1}$	$\mathbf{t_1}$	$\mathcal{T}$
$\mathbf{T_2}$	$\mathbf{f_4}$	$\mathcal{T}$	$\mathbf{t_2}$	$\mathbf{f_1}$	$\mathcal{T}$	$\mathbf{f_2}$	$\mathbf{T_4}$	$\mathcal{T}$	$\mathbf{F_2}$	$\mathbf{T_1}$	$\mathcal{T}$
$\mathbf{T_3}$	$\mathbf{F_4}$	$\mathcal{F}$	$\mathbf{t_3}$	$\mathbf{F_1}$	$\mathcal{F}$	$\mathbf{f_3}$	$\mathbf{t_4}$	$\mathcal{F}$	$\mathbf{F_3}$	$\mathbf{t_1}$	$\mathcal{F}$
$\mathbf{T_4}$	$\mathbf{f_4}$	$\mathcal{F}$	$\mathbf{t_4}$	$\mathbf{f_1}$	$\mathcal{F}$	$\mathbf{f_4}$	$\mathbf{T_4}$	$\mathcal{F}$	$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathcal{F}$

$A \tilde{\to} B$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	t_{13}	t_{24}	f_{13}	f_{24}	$\mathcal{F}_{13}$	$\mathcal{F}_{24}$
$\mathcal{T}_{13}, \mathcal{T}_{24}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	t_{13}	t_{24}	f_{13}	f_{24}	$\mathcal{F}_{13}$	$\mathcal{F}_{24}$
t_{13}, t_{24}	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	f_{13}	f_{13}	$f_{13}, \mathcal{F}_{13}$	$f_{13}, \mathcal{F}_{13}$
f_{13}, f_{24}	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	t_{13}	t_{24}	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	t_{13}	t_{24}
$\mathcal{F}_{13}, \mathcal{F}_{24}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$

where

- $\mathcal{T}_{13} = \{\mathbf{T_1}, \mathbf{T_3}\}$, $\mathcal{T}_{24} = \{\mathbf{T_2}, \mathbf{T_4}\}$, $t_{13} = \{\mathbf{t_1}, \mathbf{t_3}\}$, $t_{24} = \{\mathbf{t_2}, \mathbf{t_4}\}$,
- $\mathcal{F}_{13} = \{\mathbf{F_1}, \mathbf{F_3}\}$, $\mathcal{F}_{24} = \{\mathbf{F_2}, \mathbf{F_4}\}$, $f_{13} = \{\mathbf{f_1}, \mathbf{f_3}\}$, $f_{24} = \{\mathbf{f_2}, \mathbf{f_4}\}$,
- $\mathcal{F} = \{\mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}, \mathbf{f_4}, \mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}, \mathbf{F_4}\}$.

A $\mathsf{k_f}$-*valuation* in a $\mathsf{K_f}$-Nmatrix M is a function $v : \mathsf{Form} \to \mathcal{V}$ that satisfies the following condition for every n-ary connective $*$ of $\mathcal{L}$ and $A_1, \ldots, A_n \in \mathsf{Form}$: $v(*(A_1, \ldots, A_n)) \in \tilde{*}(v(A_1), \ldots, v(A_n))$.[11]

Remark 2.8 Note that the above truth table for $\tilde{\to}$ is making use of an unusual abbreviation. The full version is available in the Appendix.

Remark 2.9 This truth-table for implication can be seen as a generalization of the truth-table for implication of the system K, presented in [16, Definition 43]. The similarities will become explicit in the definition of the canonical model (cf. Definition 3.8 below).

Definition 2.10 A is a $\mathsf{k_f}$-*consequence* of Γ ($\Gamma \models_{\mathsf{k_f}} A$) iff for all $\mathsf{k_f}$-valuation v, if $v(B) \in \mathcal{T}$ for all $B \in \Gamma$ then $v(A) \in \mathcal{T}$. In particular, A is a $\mathsf{k_f}$-*tautology* iff $v(A) \in \mathcal{T}$ for all $\mathsf{k_f}$-valuations v.

[11] Note that the definition of $\mathsf{k_f}$-valuations is in the terminology of [16] called legal valuation, which in turn is also called dynamic valuation in [1].

Remark 2.11 Note that with Yuri V. Ivlev another logician considered non-deterministic semantics for a language with modality (cf. [10,11]). He is, however, not dealing with normal modal logics but only fragments without the rule of necessitation. Our system $\mathsf{k_f}$ can therefore also be understood as a system of modality in the sense of Ivlev. [12]

Definition 2.12 Let v be a function $v : \mathsf{Form} \to \mathcal{V}$. Then,

- v is a *0th-level* $\mathsf{K_f}$*-valuation* if v is a $\mathsf{k_f}$-valuation.
- v is a $m+1$*st-level* $\mathsf{K_f}$*-valuation* iff v is an mth-level $\mathsf{K_f}$-valuation and for every sentence A, $v(A) \in \{\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}, \mathbf{T_4}\}$ holds if $v'(A) \in \mathcal{T}$ for every mth-level $\mathsf{K_f}$-valuation v'.

Based on these, we define v to be a $\mathsf{K_f}$*-valuation* iff v is an mth-level $\mathsf{K_f}$-valuation for every $m \geq 0$.

Definition 2.13 A is a $\mathsf{K_f}$*-tautology* ($\models_{\mathsf{K_f}} A$) iff $v(A) \in \{\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}, \mathbf{T_4}\}$ for every $\mathsf{K_f}$-valuation v.

Remark 2.14 The definition of $\mathsf{K_f}$-valuations involves some complicated construction. Hence, for the sake of simplicity, we will only focus on tautologies, but not consequence relations with possibly non-empty premises for $\mathsf{K_f}$, and similarly for its extensions.

3 Soundness and completeness

We first prove the soundness, and then turn to the completeness result for both $\mathsf{k_f}$ and $\mathsf{K_f}$. The proofs are variants of those in [16].

3.1 Soundness

The soundness for the $\mathsf{k_f}$ consequence relation is rather straightforward.

Proposition 3.1 *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \vdash_{\mathsf{k_f}} A$ *then* $\Gamma \models_{\mathsf{k_f}} A$*.*

Proof. It suffices to check that all axioms are $\mathsf{k_f}$-tautologies, and that the rule of inference (MP) preserves the designated values. The details are spelled out in the Appendix. □

For the soundness of $\mathsf{K_f}$, we need the following lemma.

Lemma 3.2 *Assume that* $\vdash_{\mathsf{K_f}} A$ *and that the length of the proof for* A *is* m*. Then, for every* m*th-level* $\mathsf{K_f}$*-valuation* v*,* $v(A) \in \mathcal{T}$*.*

Proof. By induction on the length m of the proof for $\vdash_{\mathsf{K_f}} A$. For the base, case in which $m = 1$, A is one of the axioms. Since axioms are $\mathsf{k_f}$-tautologies, as shown above, $v(A) \in \mathcal{T}$ for every 1st-level $\mathsf{K_f}$-valuation. (Note that by definition, if a sentence is designated for every mth-level $\mathsf{K_f}$-valuation, then it is also designated for every $m+1$st-level $\mathsf{K_f}$-valuation.) For the induction step, assume that the result holds for proofs of the length m, and let $B_1, \ldots, B_m, B_{m+1}(= A)$ be the proof for A. Then, there are the following three cases:

[12] For continuations of Ivlev's approach, see for example [3,4,5]. For a little problem with Ivlev's original paper, see [16, §3.3].

- If A is an axiom, then A is designated for every $\mathsf{k_f}$-valuation, and thus for every $m+1$st-level $\mathsf{K_f}$-valuation as well.
- If A is obtained by applying (MP) to B_i and $B_j (= B_i \to A)$, then by induction hypothesis, B_i and B_j are designated for every $\max\{i, j\}$th-level $\mathsf{K_f}$-valuation, and thus for every mth-level $\mathsf{K_f}$-valuation. By the truth table for $\to$, we obtain that A is also designated for every mth-level $\mathsf{K_f}$-valuation. Therefore, A is designated for every $m+1$st-level $\mathsf{K_f}$-valuation as well.
- If A is obtained by applying (RN) to B_i, then by induction hypothesis, B_i is designated for every ith-level $\mathsf{K_f}$-valuation. So, for every $i+1$st-level $\mathsf{K_f}$-valuation, $\neg\Diamond\neg B_i$, i.e. A is designated. Thus, A is designated for every $m+1$st-level $\mathsf{K_f}$-valuation.

This completes the proof. □

Once we have the lemma, soundness for $\mathsf{K_f}$ follows immediately.

Proposition 3.3 *For all $A \in \mathsf{Form}$, if $\vdash_{\mathsf{K_f}} A$ then $\models_{\mathsf{K_f}} A$.*

Proof. Let the length of the proof for A be m. Then, by the above lemma, A is designated for every mth-level $\mathsf{K_f}$-valuation. Therefore, $v(A) \in \{\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}, \mathbf{T_4}\}$ for every $m+1$st-level $\mathsf{K_f}$-valuation v. Since $\mathsf{K_f}$-valuations are also $m+1$st-level $\mathsf{K_f}$-valuations, we obtain that A takes one of the values $\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}, \mathbf{T_4}$ for every $\mathsf{K_f}$-valuation, as desired. □

3.2 Completeness

We now turn to prove the completeness. First, we list some provable formulas, without proofs, that will be used in the following proofs.

Proposition 3.4 *The following formulas are provable in $\mathsf{k_f}$:*

$$\neg\Diamond\neg A \to (\Diamond\neg B \to \Diamond\neg(A \to B)) \quad (1) \qquad (A \to B) \to ((\neg A \to B) \to B) \quad (3)$$

$$A \to (\neg B \to \neg(A \to B)) \quad (2) \qquad A \to (\neg A \to B) \quad (4)$$

Second, we introduce some standard notions that will be used in the proofs. In what follows, we let L be $\mathsf{k_f}$ or $\mathsf{K_f}$, or their extensions we consider in later sections.

Definition 3.5 For $\Gamma \subseteq \mathsf{Form}$, Γ is an L-*consistent* set iff $\Gamma \nvdash A$ or $\Gamma \nvdash \neg A$ for all $A \in \mathsf{Form}$. Γ is L-*inconsistent* otherwise.

Definition 3.6 For $\Gamma \subseteq \mathsf{Form}$, Γ is *maximal* L-*consistent* set iff Γ is L-consistent and any set of formulas properly containing Γ is L-inconsistent. If Γ is maximal L-consistent set, then we say that Γ is a L-$\mathtt{mcs}$.

We then obtain the following well-known lemma. As the proof is standard, we will leave it to the reader.

Lemma 3.7 *For any $\Sigma \cup \{A\} \subseteq \mathsf{Form}$, suppose that $\Sigma \nvdash_{\mathsf{L}} A$. Then, there is a $\Pi \supseteq \Sigma$ such that Π is a L-$\mathtt{mcs}$.*

We next define the canonical valuation. This will also give us an intuitive reading of the sixteen values.

Definition 3.8 For any $\Sigma \subseteq \mathsf{Form}$, we define a function $v_\Sigma : \mathsf{Form} \to \mathcal{V}$ as follows.

$$v_\Sigma(B) := \begin{cases} \mathbf{T_1} & \text{if } \Sigma \vdash \neg\Diamond\neg B \text{ and } \Sigma \vdash B \text{ and } \Sigma \vdash \Diamond B \text{ and } \Sigma \vdash \Diamond\neg\neg B \\ \mathbf{T_2} & \text{if } \Sigma \vdash \neg\Diamond\neg B \text{ and } \Sigma \vdash B \text{ and } \Sigma \vdash \Diamond B \text{ and } \Sigma \nvdash \Diamond\neg\neg B \\ \mathbf{T_3} & \text{if } \Sigma \vdash \neg\Diamond\neg B \text{ and } \Sigma \vdash B \text{ and } \Sigma \nvdash \Diamond B \text{ and } \Sigma \vdash \Diamond\neg\neg B \\ \mathbf{T_4} & \text{if } \Sigma \vdash \neg\Diamond\neg B \text{ and } \Sigma \vdash B \text{ and } \Sigma \nvdash \Diamond B \text{ and } \Sigma \nvdash \Diamond\neg\neg B \\ \mathbf{t_1} & \text{if } \Sigma \nvdash \neg\Diamond\neg B \text{ and } \Sigma \vdash B \text{ and } \Sigma \vdash \Diamond B \text{ and } \Sigma \vdash \Diamond\neg\neg B \\ \mathbf{t_2} & \text{if } \Sigma \nvdash \neg\Diamond\neg B \text{ and } \Sigma \vdash B \text{ and } \Sigma \vdash \Diamond B \text{ and } \Sigma \nvdash \Diamond\neg\neg B \\ \mathbf{t_3} & \text{if } \Sigma \nvdash \neg\Diamond\neg B \text{ and } \Sigma \vdash B \text{ and } \Sigma \nvdash \Diamond B \text{ and } \Sigma \vdash \Diamond\neg\neg B \\ \mathbf{t_4} & \text{if } \Sigma \nvdash \neg\Diamond\neg B \text{ and } \Sigma \vdash B \text{ and } \Sigma \nvdash \Diamond B \text{ and } \Sigma \nvdash \Diamond\neg\neg B \\ \mathbf{f_1} & \text{if } \Sigma \vdash \neg\Diamond\neg B \text{ and } \Sigma \nvdash B \text{ and } \Sigma \vdash \Diamond B \text{ and } \Sigma \vdash \Diamond\neg\neg B \\ \mathbf{f_2} & \text{if } \Sigma \vdash \neg\Diamond\neg B \text{ and } \Sigma \nvdash B \text{ and } \Sigma \vdash \Diamond B \text{ and } \Sigma \nvdash \Diamond\neg\neg B \\ \mathbf{f_3} & \text{if } \Sigma \vdash \neg\Diamond\neg B \text{ and } \Sigma \nvdash B \text{ and } \Sigma \nvdash \Diamond B \text{ and } \Sigma \vdash \Diamond\neg\neg B \\ \mathbf{f_4} & \text{if } \Sigma \vdash \neg\Diamond\neg B \text{ and } \Sigma \nvdash B \text{ and } \Sigma \nvdash \Diamond B \text{ and } \Sigma \nvdash \Diamond\neg\neg B \\ \mathbf{F_1} & \text{if } \Sigma \nvdash \neg\Diamond\neg B \text{ and } \Sigma \nvdash B \text{ and } \Sigma \vdash \Diamond B \text{ and } \Sigma \vdash \Diamond\neg\neg B \\ \mathbf{F_2} & \text{if } \Sigma \nvdash \neg\Diamond\neg B \text{ and } \Sigma \nvdash B \text{ and } \Sigma \vdash \Diamond B \text{ and } \Sigma \nvdash \Diamond\neg\neg B \\ \mathbf{F_3} & \text{if } \Sigma \nvdash \neg\Diamond\neg B \text{ and } \Sigma \nvdash B \text{ and } \Sigma \nvdash \Diamond B \text{ and } \Sigma \vdash \Diamond\neg\neg B \\ \mathbf{F_4} & \text{if } \Sigma \nvdash \neg\Diamond\neg B \text{ and } \Sigma \nvdash B \text{ and } \Sigma \nvdash \Diamond B \text{ and } \Sigma \nvdash \Diamond\neg\neg B \end{cases}$$

Remark 3.9 Compared to the definition of the canonical valuation for K in [16, Lemma 52], the number of truth values has doubled, since we are treating $\Diamond B$ and $\Diamond\neg\neg B$ separately.

Lemma 3.10 *If Σ is a $\mathsf{k_f}$-$\mathtt{mcs}$, then v_Σ is a well-defined $\mathsf{k_f}$-valuation.*

Proof. The details are spelled out in the Appendix. □

Remark 3.11 By a careful examination, we also obtain that if Σ is a $\mathsf{K_f}$-$\mathtt{mcs}$, then v_Σ is a well-defined $\mathsf{k_f}$-valuation.

Based on these, we are now ready to prove the completeness of $\mathsf{k_f}$.

Theorem 3.12 *For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, if $\Gamma \models_{\mathsf{k_f}} A$ then $\Gamma \vdash_{\mathsf{k_f}} A$.*

Proof. Suppose that $\Gamma \nvdash_{\mathsf{k_f}} A$. Then by Lemma 3.7, we can construct a $\mathsf{k_f}$-$\mathtt{mcs}$ Σ_0 such that $\Gamma \subseteq \Sigma_0$. In view of Lemma 3.10, we can define a $\mathsf{k_f}$-valuation v_{Σ_0} such that $v_{\Sigma_0}(B) \in \mathcal{T}$ for every $B \in \Gamma$ and $v_{\Sigma_0}(A) \notin \mathcal{T}$. Thus we have $\Gamma \not\models_{\mathsf{k_f}} A$, as desired. □

For the completeness of $\mathsf{K_f}$, we need one more lemma.

Lemma 3.13 *Let Γ be a $\mathsf{K_f}$-$\mathtt{mcs}$. If v_Γ is a $\mathsf{k_f}$-valuation, then v_Γ is also an mth-level $\mathsf{K_f}$-valuation for every $m \geq 1$, and thus a $\mathsf{K_f}$-valuation.*

Proof. By induction on m. For the base case, we prove that v_Γ is 1st-level $\mathsf{K_f}$-valuation. Let A be a sentence that is designated for every $\mathsf{k_f}$-valuation. Assume, for reductio, that $\nvdash_{\mathsf{K_f}} A$. Then, in view of Remark 2.2, $\nvdash_{\mathsf{k_f}} A$. Now, by Lemma 3.7, there is a $\mathsf{k_f}$-$\mathtt{mcs}$ Σ such that $\Sigma \nvdash_{\mathsf{k_f}} A$. Now let v_Σ be the $\mathsf{k_f}$-valuation generated by Σ. By the definition of v_Σ, we have that $\Sigma \nvdash_{\mathsf{k_f}} A$, i.e. $v(A) \notin \mathcal{T}$. But this contradicts our assumption that A is designated for every $\mathsf{k_f}$-valuation. Therefore, we have proved that $\vdash_{\mathsf{K_f}} A$. Then by (RN), we obtain $\vdash_{\mathsf{K_f}} \neg\Diamond\neg A$. And by the definition of v_Γ, we obtain that $v_\Gamma(A) \in \{\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}, \mathbf{T_4}\}$, as desired.

For the induction step, assume that v_Γ is an mth-level $\mathsf{K_f}$-valuation, and let A be a sentence that is designated for every mth-level $\mathsf{K_f}$-valuation. Assume, for contradiction, that $\nvdash_{\mathsf{K_f}} A$. Again, in view of Remark 2.2, we obtain $\nvdash_{\mathsf{k_f}} A$. Now by Lemma 3.7, there is a $\mathsf{k_f}$-$\mathtt{mcs}$ Δ such that $\Delta \nvdash_{\mathsf{k_f}} A$. Now let v_Δ be the $\mathsf{k_f}$-valuation generated by Δ. By induction hypothesis, we have that v_Δ is an mth-level $\mathsf{K_f}$-valuation. Moreover, by the definition of v_Δ, we have that $\Delta \nvdash_{\mathsf{k_f}} A$, i.e. $v_\Delta(A) \notin \mathcal{T}$. But this contradicts our assumption that A is designated for every mth-level $\mathsf{K_f}$-valuation. Therefore, we have proved that $\vdash_{\mathsf{K_f}} A$. Then by (RN), we obtain $\vdash_{\mathsf{K_f}} \neg\Diamond\neg A$. And by the definition of v_Γ, we obtain that $v_\Gamma(A) \in \{\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}, \mathbf{T_4}\}$, as desired. □

We are now ready to prove completeness for $\mathsf{K_f}$.

Theorem 3.14 *For all $A \in \mathsf{Form}$, if $\models_{\mathsf{K_f}} A$ then $\vdash_{\mathsf{K_f}} A$.*

Proof. Suppose that $\nvdash_{\mathsf{K_f}} A$. Then by Lemma 3.7, we have an $\mathsf{K_f}$-$\mathtt{mcs}$ Σ_0 such that $\Sigma_0 \nvdash_{\mathsf{K_f}} A$. In view of Remark 3.11, we can define a $\mathsf{k_f}$-valuation v_{Σ_0}, and by Lemma 3.13, this v_{Σ_0} is also a $\mathsf{K_f}$-valuation. Since we have $v_{\Sigma_0}(A)\notin\mathcal{T}$, it is also the case that $v_{\Sigma_0}(A)\notin\{\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}, \mathbf{T_4}\}$ (since $v_{\Sigma_0}(A)\in\{\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}, \mathbf{T_4}\}$ implies that $v_{\Sigma_0}(A)\in\mathcal{T}$). Thus we obtain $\not\models_{\mathsf{K_f}} A$. □

4 On extensions of failed K

We now have a semantics for $\mathsf{K_f}$, and with this semantics at hand, we can turn to discuss an open problem left by Humberstone in [9]. Let us first explain the problem, and then outline our approach to the problem.

In [9, p.401], Humberstone pointed out that he was not successful in finding an argument, possibly a variation of the above counter-model we reviewed in Definition 2.3, establishing the unprovability of $\Diamond\neg\neg p \to \Diamond p$. Hence this problem was left open (see also [9, p.402]).

Note here that we can easily check that the Humberstone's four-valued semantics verifies both $A{\to}\Diamond A$ and $\Diamond A{\to}\neg\Diamond\neg\Diamond A$, namely axioms for $\mathsf{S5}$. This implies that the unprovability of $\Diamond\neg\neg p \to \Diamond p$ is *not* due to the weakness of $\mathsf{K_f}$. In other words, the warning of choice of primitives carries over to extensions of $\mathsf{K_f}$, as well.

Based on this observation, we will mainly focus on extensions of $\mathsf{K_f}$ obtained by adding axioms for $\mathsf{S5}$, and analyse the open problem of Humberstone in some detail. More specifically, we not only establish the unprovability of $\Diamond\neg\neg p \to \Diamond p$, but also offer sound and complete semantics for extensions obtained by adding one or both of $\Diamond A \to \Diamond\neg\neg A$ and $\Diamond\neg\neg A \to \Diamond A$. In order to show how the number of truth values will be reduced, we will also introduce an extension of $\mathsf{K_f}$ obtained by adding an axiom for T.

4.1 From failed K to failed T

First, the extensions of $\mathsf{K_f}$ and $\mathsf{k_f}$ are obtained as follows.

Definition 4.1 The systems $\mathsf{T_f}$ and $\mathsf{t_f}$ are obtained by adding $A{\to}\Diamond A$ to $\mathsf{K_f}$ and $\mathsf{k_f}$, respectively. The consequence relations are defined as in Definition 2.1.

For the semantics, we introduce the following Nmatrix obtained by eliminating truth-values of the $\mathsf{K_f}$-Nmatrix.

Definition 4.2 A $\mathsf{T_f}$-*Nmatrix* for $\mathcal{L}$ is a tuple $M = \langle \mathcal{V}, \mathcal{T}, \mathcal{O} \rangle$, where:

(a) $\mathcal{V} = \{\mathbf{T_1}, \mathbf{t_1}, \mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}, \mathbf{F_4}\}$,

(b) $\mathcal{T} = \{\mathbf{T_1}, \mathbf{t_1}\}$,

(c) For every n-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{*}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} \setminus \{\emptyset\}$ as follows (we omit the brackets for sets):

A	$\tilde{\neg} A$	$\Diamond A$	$A \tilde{\to} B$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{F_2}$	$\mathbf{F_3}$	$\mathbf{F_4}$
$\mathbf{T_1}$	$\mathbf{F_4}$	$\mathcal{T}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathcal{F}_{13}$	$\mathcal{F}_{24}$	$\mathcal{F}_{13}$	$\mathcal{F}_{24}$
$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathcal{T}$	$\mathbf{t_1}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{F}_{13}$	$\mathcal{F}_{13}$	$\mathcal{F}_{13}$	$\mathcal{F}_{13}$
$\mathbf{F_1}$	$\mathbf{t_1}$	$\mathcal{T}$	$\mathbf{F_1}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$
$\mathbf{F_2}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathbf{F_2}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$
$\mathbf{F_3}$	$\mathbf{t_1}$	$\mathcal{F}$	$\mathbf{F_3}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$
$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathcal{F}$	$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$

where $\mathcal{F}_{13} = \{\mathbf{F_1}, \mathbf{F_3}\}$, $\mathcal{F}_{24} = \{\mathbf{F_2}, \mathbf{F_4}\}$ and $\mathcal{F} = \{\mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}, \mathbf{F_4}\}$.

Remark 4.3 The definitions of $\mathsf{t_f}$-valuations, mth-level valuations and consequence relations are exactly as in Definitions 2.7, 2.10, 2.12 and 2.13, with the difference that only the value $\mathbf{T}_1$ is preserved in the hierarchy, respectively.

Proposition 4.4 (Soundness) *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *(i) if* $\Gamma \vdash_{\mathsf{t_f}} A$ *then* $\Gamma \models_{\mathsf{t_f}} A$, *and (ii) if* $\vdash_{\mathsf{T_f}} A$ *then* $\models_{\mathsf{T_f}} A$.

Proof. The proof is similar to the proof for Proposition 3.3. □

For the completeness, we need the following definition and lemma as before.

Definition 4.5 For any $\Sigma \subseteq \mathsf{Form}$, we define a function $v_\Sigma : \mathsf{Form} \to \mathcal{V}$ as follows.

$$v_\Sigma(B) := \begin{cases} \mathbf{T_1} & \text{if } \Sigma \vdash \neg\Diamond\neg B \text{ and } \Sigma \vdash B \text{ and } \Sigma \vdash \Diamond B \text{ and } \Sigma \vdash \Diamond\neg\neg B \\ \mathbf{t_1} & \text{if } \Sigma \nvdash \neg\Diamond\neg B \text{ and } \Sigma \vdash B \text{ and } \Sigma \vdash \Diamond B \text{ and } \Sigma \vdash \Diamond\neg\neg B \\ \mathbf{F_1} & \text{if } \Sigma \nvdash \neg\Diamond\neg B \text{ and } \Sigma \nvdash B \text{ and } \Sigma \vdash \Diamond B \text{ and } \Sigma \vdash \Diamond\neg\neg B \\ \mathbf{F_2} & \text{if } \Sigma \nvdash \neg\Diamond\neg B \text{ and } \Sigma \nvdash B \text{ and } \Sigma \vdash \Diamond B \text{ and } \Sigma \nvdash \Diamond\neg\neg B \\ \mathbf{F_3} & \text{if } \Sigma \nvdash \neg\Diamond\neg B \text{ and } \Sigma \nvdash B \text{ and } \Sigma \nvdash \Diamond B \text{ and } \Sigma \vdash \Diamond\neg\neg B \\ \mathbf{F_4} & \text{if } \Sigma \nvdash \neg\Diamond\neg B \text{ and } \Sigma \nvdash B \text{ and } \Sigma \nvdash \Diamond B \text{ and } \Sigma \nvdash \Diamond\neg\neg B \end{cases}$$

Lemma 4.6 *If* Σ *is a* $\mathsf{t_f}$-$\mathtt{mcs}$, *then,* v_Σ *is a well-defined* $\mathsf{t_f}$-*valuation.*

Proof. The details of the proof are exactly as in Lemma 3.10, except that we eliminate the values $\mathbf{T_2}$, $\mathbf{T_3}$, $\mathbf{T_4}$, $\mathbf{t_2}$, $\mathbf{t_3}$, $\mathbf{t_4}$, $\mathbf{f_1}$, $\mathbf{f_2}$, $\mathbf{f_3}$ and $\mathbf{f_4}$. □

Now we can prove the completeness.

Theorem 4.7 (Completeness) *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *(i) if* $\Gamma \models_{\mathsf{t_f}} A$ *then* $\Gamma \vdash_{\mathsf{t_f}} A$, *and (ii) if* $\models_{\mathsf{T_f}} A$ *then* $\vdash_{\mathsf{T_f}} A$.

Proof. Similar to the proofs of Theorems 3.12 and 3.14, by making use of Lemma 4.6 instead of Lemma 3.10. We leave the details to the reader. $\square$

4.2 From failed T to failed $\mathsf{S5}$

We now turn to the failed $\mathsf{S5}$ which will serve as the basic system in analyzing Humberstone's open problem. For the proof system, we add three more axioms.

Definition 4.8 The systems $\mathsf{S5_f}$ and $\mathsf{s5_f}$ are obtained by adding $\Diamond\Diamond A \to \Diamond A$, $\Diamond\neg\neg\Diamond A \to \Diamond A$ and $\Diamond A \to \neg\Diamond\neg\Diamond A$ to $\mathsf{T_f}$ and $\mathsf{t_f}$, respectively. The consequence relations $\vdash_{\mathsf{S5_f}}$ and $\vdash_{\mathsf{s5_f}}$ are defined as in Definition 2.1.

For the semantics, the number of truth values will remain the same, but we eliminate non-determinacy for the truth-tables of $\Diamond$.

Definition 4.9 An $\mathsf{S5_f}$-*Nmatrix* for $\mathcal{L}$ is a tuple $M = \langle \mathcal{V}, \mathcal{T}, \mathcal{O} \rangle$, where:

(a) $\mathcal{V} = \{\mathbf{T_1}, \mathbf{t_1}, \mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}, \mathbf{F_4}\}$,

(b) $\mathcal{T} = \{\mathbf{T_1}, \mathbf{t_1}\}$,

(c) For every n-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{*}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} \setminus \{\emptyset\}$ as follows (we omit the brackets for sets):

A	$\tilde{\neg}A$	$\Diamond A$	$A \tilde{\to} B$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{F_2}$	$\mathbf{F_3}$	$\mathbf{F_4}$
$\mathbf{T_1}$	$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathcal{F}_{13}$	$\mathcal{F}_{24}$	$\mathcal{F}_{13}$	$\mathcal{F}_{24}$
$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{F}_{13}$	$\mathcal{F}_{13}$	$\mathcal{F}_{13}$	$\mathcal{F}_{13}$
$\mathbf{F_1}$	$\mathbf{t_1}$	$\mathbf{T_1}$	$\mathbf{F_1}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$
$\mathbf{F_2}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{F_2}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$
$\mathbf{F_3}$	$\mathbf{t_1}$	$\mathbf{F_4}$	$\mathbf{F_3}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$
$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathbf{F_4}$	$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$

where $\mathcal{F}_{13} = \{\mathbf{F_1}, \mathbf{F_3}\}$ and $\mathcal{F}_{24} = \{\mathbf{F_2}, \mathbf{F_4}\}$.

Remark 4.10 The definitions of $\mathsf{s5_f}$ valuations, mth-level valuations and consequence relations are exactly as in the Definitions 2.7, 2.10, 2.12 and 2.13.

There is, however, a significant property of $\mathsf{S5_f}$ which distinguishes it from the other systems introduced in this article so far. Indeed, we do not need a whole hierarchy of mth-level valuations, but only two levels are sufficient. We leave the details to the interested reader and refer to [16, §4.4] in which this was observed for $\mathsf{S4}$ and $\mathsf{S5}$.

Proposition 4.11 (Soundness) *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *(i) if* $\Gamma \vdash_{\mathsf{s5_f}} A$ *then* $\Gamma \models_{\mathsf{s5_f}} A$, *and (ii) if* $\vdash_{\mathsf{S5_f}} A$ *then* $\models_{\mathsf{S5_f}} A$.

Proof. We only note that the additional axioms are valid in the $\mathsf{S5_f}$-Nmatrix in which all non-determinacies for $\Diamond$ are eliminated. The rest is exactly as in Proposition 4.4. $\square$

Theorem 4.12 (Completeness) *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *(i) if* $\Gamma \models_{\mathsf{s5_f}} A$ *then* $\Gamma \vdash_{\mathsf{s5_f}} A$, *and (ii) if* $\models_{\mathsf{S5_f}} A$ *then* $\vdash_{\mathsf{S5_f}} A$.

Proof. We only note that the additional axioms allow us to conclude that if Σ is a $\mathsf{s5_f}$-$\mathtt{mcs}$, then v_Σ is a well-defined $\mathsf{s5_f}$-valuation. The proof is by induction, and we only check the case when B is of the form $\Diamond C$.

- If $v_\Sigma(C) = \mathbf{T_1}$, then by IH, we obtain that $\Sigma \vdash \neg\Diamond\neg C$ and $\Sigma \vdash C$ and $\Sigma \vdash \Diamond C$ and $\Sigma \vdash \Diamond\neg\neg C$. Then, by the third conjunct and $\Diamond A \to \neg\Diamond\neg\Diamond A$, we have $\Sigma \vdash \neg\Diamond\neg\Diamond C$, and this also implies $\Sigma \vdash \Diamond C$, $\Sigma \vdash \Diamond\Diamond C$ and $\Sigma \vdash \Diamond\neg\neg\Diamond C$. Then, by the definition of v_Σ, this means $v_\Sigma(\Diamond C) = \mathbf{T_1}$, as desired. Other cases with $v_\Sigma(C){=}\mathbf{t_1}$, $v_\Sigma(C){=}\mathbf{F_1}$ and $v_\Sigma(C){=}\mathbf{F_2}$ are the same.
- If $v_\Sigma(C) = \mathbf{F_3}$, then by IH, we obtain that $\Sigma \not\vdash \neg\Diamond\neg C$ and $\Sigma \not\vdash C$ and $\Sigma \not\vdash \Diamond C$ and $\Sigma \vdash \Diamond\neg\neg C$. Then, by the third conjunct together with $\Diamond\Diamond A \to \Diamond A$ and $\Diamond\neg\neg\Diamond A \to \Diamond A$, we have $\Sigma \not\vdash \Diamond\Diamond C$ and $\Sigma \vdash \Diamond\neg\neg\Diamond C$, and we also have $\Sigma \not\vdash \neg\Diamond\neg\Diamond C$ and $\Sigma \not\vdash \Diamond C$. Then, by the definition of v_Σ, this means $v_\Sigma(\Diamond C) = \mathbf{F_4}$, as desired. The other case with $v_\Sigma(C){=}\mathbf{F_4}$ is the same.

The rest of the proof is exactly as in Theorem 4.7. □

4.3 On the open problem of Humberstone

We are now in the position to shift our interest to the problem left open by Humberstone in [9]. The counter-model given by Humberstone, and described in §2.2, invalidates one direction of the equivalence $\Diamond A \leftrightarrow \Diamond\neg\neg A$, namely $\Diamond A \to \Diamond\neg\neg A$, while validating the other. We will now show that the above semantics for $\mathsf{S5_f}$ in the style of Kearns, with one more adjustment, helps us to establish the unprovability of both $\Diamond p \to \Diamond\neg\neg p$ and $\Diamond\neg\neg p \to \Diamond p$.

The adjustment we need to make is, to close the non-determinacies to obtain a six-valued (deterministic) matrix that will do the job for our present purposes. One may have expected that the above semantics will directly give us the counter-model. Unfortunately, this is not the case, at least at the moment, due to the problem of analyticity in Kearns' semantics (cf. [16, Remark 42]).[13] Still, the definition of the canonical valuation strongly suggests that we should be able to give a counter-model, and by following Schiller Joe Scroggs who explored the many-valued extensions of $\mathsf{S5}$ in [17], we may aim at a deterministic extension of our semantics for $\mathsf{S5_f}$.

Definition 4.13 A $\mathsf{dS5_f}$-*matrix* for $\mathcal{L}$ is a tuple $M = \langle \mathcal{V}, \mathcal{T}, \mathcal{O} \rangle$, where:

(a) $\mathcal{V} = \{\mathbf{T_1}, \mathbf{t_1}, \mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}, \mathbf{F_4}\}$,

(b) $\mathcal{T} = \{\mathbf{T_1}\}$,

(c) For every n-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{*}$ from $\mathcal{V}^n$ to $\mathcal{V}$ as follows:

A	$\tilde{\neg}A$	$\Diamond A$	$A \tilde{\to} B$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{F_2}$	$\mathbf{F_3}$	$\mathbf{F_4}$
$\mathbf{T_1}$	$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{F_4}$	$\mathbf{F_1}$	$\mathbf{F_4}$
$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{F_1}$	$\mathbf{F_1}$	$\mathbf{F_1}$	$\mathbf{F_1}$
$\mathbf{F_1}$	$\mathbf{t_1}$	$\mathbf{T_1}$	$\mathbf{F_1}$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{T_1}$	$\mathbf{t_1}$
$\mathbf{F_2}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{F_2}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{T_1}$
$\mathbf{F_3}$	$\mathbf{t_1}$	$\mathbf{F_4}$	$\mathbf{F_3}$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{T_1}$	$\mathbf{t_1}$
$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathbf{F_4}$	$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{T_1}$

[13] It is not yet proven that in Kearns' semantics a partial valuation that falsifies a formula can be extended to a full valuation that necessarily falsifies the formula, as well. See also [1] for a discussion on analyticity related to non-deterministic semantics in general.

We refer to the semantic consequence relation defined in terms of preservation of the designated value with the above matrix as $\models_{\mathsf{dS5}}$.

Then, it is tedious but routine to check the following.

Lemma 4.14 *For all $A \in \mathsf{Form}$, if $\vdash_{\mathsf{S5_f}} A$ then $\models_{\mathsf{dS5}} A$.*

We are now ready to answer Humberstone's open problem.

Theorem 4.15 $\nvdash_{\mathsf{S5_f}} \Diamond p \to \Diamond\neg\neg p$ *and* $\nvdash_{\mathsf{S5_f}} \Diamond\neg\neg p \to \Diamond p$.

Proof. In view of the above lemma, it suffices to check that $\not\models_{\mathsf{dS5}} \Diamond p \to \Diamond\neg\neg p$ and $\not\models_{\mathsf{dS5}} \Diamond\neg\neg p \to \Diamond p$. For the first item, assign $\mathbf{F_2}$ to A of the $\mathsf{dS5_f}$-matrix. Then, $\Diamond p$ receives the value $\mathbf{T_1}$ and $\Diamond\neg\neg p$ receives the value $\mathbf{F_4}$. Therefore, $\Diamond p \to \Diamond\neg\neg p$ receives the non-designated value $\mathbf{F_4}$, as desired. For the second item, assign $\mathbf{F_3}$ to p of the $\mathsf{dS5_f}$-matrix. Then, $\Diamond\neg\neg p$ receives the value $\mathbf{T_1}$ and $\Diamond p$ receives the value $\mathbf{F_4}$. Therefore, $\Diamond\neg\neg p \to \Diamond p$ receives the non-designated value $\mathbf{F_4}$, as desired. □

Remark 4.16 In view of the definition of the canonical valuation, this result is of course something expected. The emphasis here should be that some technical devices are available to make the canonical valuation work as designed, thanks to the semantic framework introduced by Kearns. This is, in turn, giving us some new insight on the problem left open by Humberstone.

Let us now continue by introducing further extensions of $\mathsf{S5_f}$ and $\mathsf{s5_f}$ obtained by adding one of the two formulas $\Diamond A \to \Diamond\neg\neg A$ and $\Diamond\neg\neg A \to \Diamond A$.

Definition 4.17 Let $\mathsf{S5_{fc}}$ and $\mathsf{s5_{fc}}$ be the systems obtained by adding $\Diamond A \to \Diamond\neg\neg A$ to $\mathsf{S5_f}$ and $\mathsf{s5_f}$, respectively. Moreover, let $\mathsf{S5_{fa}}$ and $\mathsf{s5_{fa}}$ be the systems obtained by adding $\Diamond\neg\neg A \to \Diamond A$ to $\mathsf{S5_f}$ and $\mathsf{s5_f}$, respectively.[14]

For the semantics, we need to eliminate one value each that was used to invalidate the additional axiom.

Definition 4.18 The Nmatrices for $\mathsf{S5_{fc}}$ and $\mathsf{S5_{fa}}$ are obtained from the Nmatrix for $\mathsf{S5_f}$, by eliminating the values $\mathbf{F_2}$ and $\mathbf{F_3}$, respectively. More specifically, an $\mathsf{S5_{fc}}$-*Nmatrix* for $\mathcal{L}$ is a tuple $M = \langle \mathcal{V}, \mathcal{T}, \mathcal{O} \rangle$, where:

(a) $\mathcal{V} = \{\mathbf{T_1}, \mathbf{t_1}, \mathbf{F_1}, \mathbf{F_3}, \mathbf{F_4}\}$,

(b) $\mathcal{T} = \{\mathbf{T_1}, \mathbf{t_1}\}$,

(c) For every n-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{*}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} \setminus \{\emptyset\}$ as follows (we omit the brackets for sets):

A	$\tilde{\neg} A$	$\Diamond A$
$\mathbf{T_1}$	$\mathbf{F_4}$	$\mathbf{T_1}$
$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{T_1}$
$\mathbf{F_1}$	$\mathbf{t_1}$	$\mathbf{T_1}$
$\mathbf{F_3}$	$\mathbf{t_1}$	$\mathbf{F_4}$
$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathbf{F_4}$

$A \tilde{\to} B$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{F_3}$	$\mathbf{F_4}$
$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathcal{F}_{13}$	$\mathcal{F}_{13}$	$\mathbf{F}_4$
$\mathbf{t_1}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{F}_{13}$	$\mathcal{F}_{13}$	$\mathcal{F}_{13}$
$\mathbf{F_1}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$
$\mathbf{F_3}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$
$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$

[14] Note that additional subscripts c and a are for consequent and antecedent respectively, indicating the position of the double negation in the additional axiom.

where $\mathcal{F}_{13} = \{\mathbf{F_1}, \mathbf{F_3}\}$.

Moreover, an $\mathsf{S5_{fa}}$-*Nmatrix* for $\mathcal{L}$ is a tuple $M = \langle \mathcal{V}, \mathcal{T}, \mathcal{O} \rangle$, where:

(a) $\mathcal{V} = \{\mathbf{T_1}, \mathbf{t_1}, \mathbf{F_1}, \mathbf{F_2}, \mathbf{F_4}\}$,

(b) $\mathcal{T} = \{\mathbf{T_1}, \mathbf{t_1}\}$,

(c) For every n-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{*}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} \setminus \{\emptyset\}$ as follows (we omit the brackets for sets):

A	$\tilde{\neg}A$	$\Diamond A$	$A \tilde{\to} B$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{F_2}$	$\mathbf{F_4}$
$\mathbf{T_1}$	$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{F}_1$	$\mathcal{F}_{24}$	$\mathcal{F}_{24}$
$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathbf{F}_1$	$\mathbf{F}_1$	$\mathbf{F}_1$
$\mathbf{F_1}$	$\mathbf{t_1}$	$\mathbf{T_1}$	$\mathbf{F_1}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$
$\mathbf{F_2}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{F_2}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$
$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathbf{F_4}$	$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$

where $\mathcal{F}_{24} = \{\mathbf{F_2}, \mathbf{F_4}\}$.

We can then establish soundness and completeness for the four new systems introduced in this subsection. Indeed, all definitions, propositions and theorems are exactly as in §4.1 and §4.2. More specifically, all proofs can be obtained by slightly modifying the proofs of the mentioned subsections, by removing the values $\mathbf{F_2}$ and $\mathbf{F_3}$, respectively. We safely leave the details for the interested reader.

Remark 4.19 Note that we can strengthen Theorem 4.15 as follows: $\not\models_{\mathsf{S5_{fa}}} \Diamond p \to \Diamond \neg\neg p$ and $\not\models_{\mathsf{S5_{fc}}} \Diamond \neg\neg p \to \Diamond p$. This is precisely because we can make use of the submatrices of the six-valued $\mathsf{dS5_f}$-matrix introduced in Definition 4.13.

4.4 From failed S5 to full S5

As noted by Humberstone in [9, p.401], we obtain the standard normal modal logic K if we extend $\mathsf{K_f}$ by adding $\Diamond A \leftrightarrow \Diamond \neg\neg A$ since this gives us the equivalence $\neg\neg\Diamond\neg\neg A \leftrightarrow \Diamond A$ which corresponds to the $\neg\Box\neg A \leftrightarrow \Diamond A$ used in [2, p.34]. Since this observation also carries over to extensions of K, it is rather natural to introduce the common extension of $\mathsf{S5_{fa}}$ and $\mathsf{S5_{fc}}$ obtained by adding the missing direction.

Definition 4.20 The systems $\mathsf{S5}\Diamond$ and $\mathsf{s5}\Diamond$ are obtained by adding the axiom scheme $\Diamond A \leftrightarrow \Diamond \neg\neg A$ to $\mathsf{S5_f}$ and $\mathsf{s5_f}$, respectively.

For the semantics, seen from the $\mathsf{S5_f}$-Nmatrix, we need to eliminate both values that were used to invalidate the additional axioms. Equivalently, we obtain the same Nmatrix from the $\mathsf{S5_{fa}}$-Nmatrix and the $\mathsf{S5_{fc}}$-Nmatrix by eliminating the values that we used to invalidate the missing direction.

Definition 4.21 An $\mathsf{S5}\Diamond$-*Nmatrix* for $\mathcal{L}$ is a tuple $M = \langle \mathcal{V}, \mathcal{T}, \mathcal{O} \rangle$, where:

(a) $\mathcal{V} = \{\mathbf{T_1}, \mathbf{t_1}, \mathbf{F_1}, \mathbf{F_4}\}$,

(b) $\mathcal{T} = \{\mathbf{T_1}, \mathbf{t_1}\}$,

(c) For every n-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{*}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} \setminus \{\emptyset\}$ as follows (we omit the brackets for sets):

A	$\tilde{\neg}A$	$\Diamond A$	$A\tilde{\to}B$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{F_4}$
$\mathbf{T_1}$	$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{F_4}$
$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{T_1}$	$\mathbf{T_1}, \mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{F_1}$
$\mathbf{F_1}$	$\mathbf{t_1}$	$\mathbf{T_1}$	$\mathbf{F_1}$	$\mathbf{T_1}$	$\mathbf{T_1}, \mathbf{t_1}$	$\mathbf{T_1}, \mathbf{t_1}$	$\mathbf{T_1}, \mathbf{t_1}$
$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathbf{F_4}$	$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathbf{T_1}, \mathbf{t_1}$	$\mathbf{T_1}, \mathbf{t_1}$	$\mathbf{T_1}, \mathbf{t_1}$

The rest of the details towards soundness and completeness results will be as before. Indeed, all definitions, propositions and theorems are exactly as in §4.1 and §4.2, and all proofs can be obtained by slightly modifying the proofs of the mentioned subsections, by removing both of the values $\mathbf{F_2}$ and $\mathbf{F_3}$.

Remark 4.22 A closer look at the definitions reveals that the Nmatrix of $\mathsf{S5}\Diamond$ and its definition of the canonical valuation are very similar to the ones for $\mathsf{S5}$ given in [16]. In fact, the definitions of the canonical valuations are equivalent since in view of the additional axiom, the distinction between $\Diamond B$ and $\Diamond\neg\neg B$ is redundant. For the Nmatrix, however, this is slightly different since the one in [16], taken from [12], has the following truth-table for $\tilde{\to}$.

$A\tilde{\to}B$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{F_4}$
$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{F_4}$
$\mathbf{t_1}$	$\mathbf{T_1}$	$\mathbf{T_1}, \mathbf{t_1}$	$\mathbf{F_1}$	$\mathbf{F_1}$
$\mathbf{F_1}$	$\mathbf{T_1}$	$\mathbf{T_1}, \mathbf{t_1}$	$\mathbf{T_1}, \mathbf{t_1}$	$\mathbf{t_1}$
$\mathbf{F_4}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{T_1}$	$\mathbf{T_1}$

Indeed, there are some additional non-determiniacies in our $\mathsf{S5}\Diamond$-Nmatrix:

- $\mathbf{F_1}\tilde{\to}\mathbf{F_4}$ will be $\mathbf{t_1}$ by having $\neg\Diamond\neg(A \to B) \to (\Diamond A \to \Diamond B)$ as derivable;
- $\mathbf{F_4}\tilde{\to}x$ for all $x\in\mathcal{V}$ will be always $\mathbf{T_1}$ by having $\Diamond\neg(A\to B)\to\Diamond A$ as derivable.

However, at the level of $\mathsf{S5}\Diamond$-valuations, where some valuations will be ruled out, the formulas above will be validated, and thus the two semantics, the one for $\mathsf{S5}\Diamond$ given above, and the one for $\mathsf{S5}$, introduced in [12] are indeed equivalent.

5 Concluding remarks

Let us now briefly summarize our main results of this paper, and then discuss a few items for future directions.

Main results Building on the observation of Humberstone about the choice of primitives in [9], we aimed at offering a sound and complete semantics for the failed axiomatization of the modal logic $\mathsf{K_f}$, a variant of the modal logic K with the possibility operator as the only primitive modal operator, but without $\neg\neg\Diamond\neg\neg A\leftrightarrow\Diamond A$, the key axiom to obtain an axiomatization based on $\Diamond$. The resulting semantics is based on a sixteen-valued non-deterministic semantics which can be seen as a variant of the semantics devised by Kearns in [12].

We also discussed an open problem left by Humberstone in [9], showing the independence of not only $\Diamond A \to \Diamond\neg\neg A$ but also $\Diamond\neg\neg A \to \Diamond A$ from $\mathsf{S5_f}$ (failed axiomatization of $\mathsf{S5}$), therefore also from $\mathsf{K_f}$. To this end, we devised a semantics for $\mathsf{S5_f}$ based on a six-valued non-deterministic semantics, and used a deterministic extension to establish the unprovability of both $\Diamond p \to \Diamond\neg\neg p$

and $\Diamond\neg\neg p \to \Diamond p$ in one single matrix.

The extensions of $\mathsf{K_f}$ we discussed in this article can be ordered, from left to right by deductive strength, in the following way:

$$\begin{tikzcd}[column sep=small]
& & & \mathsf{S5_{fc}} \arrow[dr, dash] & \\
\mathsf{K_f} \arrow[r, dash] & \mathsf{T_f} \arrow[r, dash] & \mathsf{S5_f} \arrow[ur, dash] \arrow[dr, dash] & & \mathsf{S5}\Diamond \\
& & & \mathsf{S5_{fa}} \arrow[ur, dash] &
\end{tikzcd}$$

Note that Humberstone also specifically asked for a counter-model for $\Diamond\neg\neg A \to \Diamond A$ that is closer to his own counter-model we revisited in Definition 2.3. There is a question of how to precisify the notion of closeness to Humberstone's counter-model, but there is one possible answer due to Xuefeng Wen presented in [23, p.71], independently of Humberstone's question. [15] More specifically, Wen's model modifies the standard model $\mathcal{M} = \langle W, R, V\rangle$ for the modal logic K by changing the usual truth condition for $\Diamond$ as follows.

$$\mathcal{M}, x \Vdash \Diamond A \text{ iff } A{=}\neg B \text{ and for some } y \in W, xRy \text{ and } \mathcal{M}, y \nVdash B.$$

Then, we may easily observe that $\Diamond\neg\neg p \to \Diamond p$ fails in a model with $W = \{w\}$, $R = \{(w, w)\}$ and $V(p) = \{w\}$. Note, however, that $\Diamond A \to \Diamond\neg\neg A$ is valid in the model suggested by Wen. We therefore leave the task to explore the exact relations between Humberstone's counter-model, Wen's counter-model and our model, for interested readers.

Future directions (I): a systematic study on extensions of failed K Since our motivation came from Humberstone's interesting observations reported in [9], we only focused on extensions of $\mathsf{K_f}$ that were crucial and helpful for our observations. However, this does not exclude a more systematic study of extensions of $\mathsf{K_f}$. We will here offer a sketch of some of the facts that seem to suggest that the landscape of the extensions of $\mathsf{K_f}$ may look quite different from the extensions of K.

First, it is well known that in considering extensions of K, there are two equivalent formulations for many cases. For example, additions of $\Box A \to A$ and $A \to \Diamond A$ will both give rise to the modal logic T. This will no longer be the case for extensions of $\mathsf{K_f}$. Indeed, as we observed in §4.1, the extension of $\mathsf{K_f}$ by $A \to \Diamond A$ was characterized by a semantics obtained by eliminating ten values from the semantics for $\mathsf{K_f}$. However, if we extend $\mathsf{K_f}$ by $\neg\Diamond\neg A \to A$, then we can only eliminate eight values.

Something similar happens to D-like systems when we add $\neg\Diamond\neg A \to \Diamond A$ and $\neg\Diamond\neg A \to \Diamond\neg\neg A$ to $\mathsf{K_f}$. More specifically, the former requires elimination of six values whereas the latter only requires to eliminate four values. Moreover, there will be a deviation from the usual picture in the sense that $\mathsf{K_f}$ plus $\neg\Diamond\neg A \to A$, a T-like system, and $\mathsf{K_f}$ plus $\neg\Diamond\neg A \to \Diamond A$, D-like system, are incomparable. Indeed, we may establish that $\neg\Diamond\neg A \to \Diamond A$ is unprovable in the first system and $\neg\Diamond\neg A \to A$ is unprovable in the second system in a similar

[15] Our sincere thanks go to one of the anonymous referees for pointing this out.

manner as we did in Theorem 4.15. We can then again order the extensions of $\mathsf{K_f}$, up to $\mathsf{T_f}$, from left to right by deductive strength, in the following way:

$$
\begin{tikzcd}
 & & \mathsf{K_f} + \neg\Diamond\neg A \to A \arrow[dr, dash] & \\
\mathsf{K_f} \arrow[r, dash] & \mathsf{K_f} + \neg\Diamond\neg A \to \Diamond\neg\neg A \arrow[ur, dash] \arrow[dr, dash] & & \mathsf{T_f} \\
 & & \mathsf{K_f} + \neg\Diamond\neg A \to \Diamond A \arrow[ur, dash] &
\end{tikzcd}
$$

Therefore, it can be safely said, the class of extensions for $\mathsf{K_f}$ looks quite different than usually for normal modal logics.

Future directions (II): Failed axiomatizations of tense logics One of the *examples from life* for failed axiomatizations, as Humberstone put it in [9], is that of tense logics. In particular, he discussed the system $\mathsf{K_t}$, given by S. K. Thomason in [22], introduced as a bimodal logic with two primitive possibility operators. It was shown by Humberstone that the axiomatization for $\mathsf{K_t}$ fails to be complete with respect to Kripke semantics, by a similar argument building on a variation of his own counter-model.

In light of the results of this article, we believe that it is possible to construct sound and complete Kearns' semantics for $\mathsf{K_t}$ or even its extensions. We have not spelled out the details, but probably an Nmatrix with 128 truth-values suffices to prove the desired result for $\mathsf{K_t}$, and for certain extensions of $\mathsf{K_t}$, the number of truth-values should be reduced significantly. The key idea for constructing such semantics, lies in the canonical model construction, where we would need to add conditions for the interaction of the two modal operators.

Future directions (III): Some critical reflections on Kearns' semantics Finally, we left out an important question raised by readers of earlier versions of this article, the question whether this semantics is of any philosophical value. At the moment we are far away from claiming any philosophical significance , unless we follow Kearns' discussion (cf. [13]). In the end, we fully agree with B. J. Copeland in [6, p. 400], when he writes:

> "Philosophically significant semantical arguments can be yielded only by philosophically significant semantics, not by merely formal model theory."

Thus, it remains as a (huge) challenge to explore if we can turn Kearns' semantics into a philosophically significant semantics.

Appendix

In this appendix, we spell out the details left open in the text and give the full truth-table of the $\mathsf{K_f}$ conditional from Definition 2.7.

Details of Proposition 3.1 We will only prove the case for (LK), since the proofs for the other modal axioms as well as the classical axioms and (MP) are similar. Now, suppose that for a $\mathsf{k_f}$-valuation $v(\neg\Diamond\neg(A\to B)\to(\neg\Diamond\neg A\to\neg\Diamond\neg B)) \notin \mathcal{T}$. Then, this implies (1) $v(\neg\Diamond\neg(A \to B)) \in \mathcal{T}$, (2) $v(\neg\Diamond\neg A) \in \mathcal{T}$ and (3) $v(\neg\Diamond\neg B) \notin \mathcal{T}$. Now, we can see that three conditions imply the following, respectively:

(1) then $v(\Diamond\neg(A \to B)) \notin \mathcal{T}$
then $v(\neg(A \to B)) \in \{\mathbf{T_3}, \mathbf{t_3}, \mathbf{T_4}, \mathbf{t_4}, \mathbf{F_3}, \mathbf{f_3}, \mathbf{F_4}, \mathbf{f_4}\}$
then $v(A \to B) \in \{\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}, \mathbf{T_4}, \mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}, \mathbf{f_4}\}$

(2) then $v(\Diamond\neg A) \notin \mathcal{T}$
then $v(\neg A) \in \{\mathbf{T_3}, \mathbf{t_3}, \mathbf{T_4}, \mathbf{t_4}, \mathbf{F_3}, \mathbf{f_3}, \mathbf{F_4}, \mathbf{f_4}\}$
then $v(A) \in \{\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}, \mathbf{T_4}, \mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}, \mathbf{f_4}\}$

(3) then $v(\Diamond\neg B) \in \mathcal{T}$
then $v(\neg B) \in \{\mathbf{T_1}, \mathbf{t_1}, \mathbf{T_2}, \mathbf{t_2}, \mathbf{F_1}, \mathbf{f_1}, \mathbf{F_2}, \mathbf{f_2}\}$
then $v(B) \in \{\mathbf{t_1}, \mathbf{t_2}, \mathbf{t_3}, \mathbf{t_4}, \mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}, \mathbf{F_4}\}$

By looking at the truth tables, this is not possible.

Proof of Lemma 3.10 Note first that v_Σ is well-defined. The desired result can be proved by induction on the number n of connectives.
(Base): For atomic formulas, this is obvious in view of the definition of v_Σ.
(Induction step): We split the cases based on the connectives.
Case 1. If $B = \neg C$, then we have sixteen cases of which we will prove four.

- If $v_\Sigma(C) \in \{\mathbf{T_2}, \mathbf{T_4}\}$, then by IH, we obtain that $\Sigma \vdash \neg\Diamond\neg C$ and $\Sigma \vdash C$ and $\Sigma \nvdash \Diamond\neg\neg C$. From this, we immediately get $\Sigma \vdash \neg\Diamond\neg\neg C$ and $\Sigma \nvdash \neg C$ and $\Sigma \nvdash \Diamond\neg C$ and $\Sigma \nvdash \Diamond\neg\neg\neg C$. Then, by the definition of v_Σ, this means $v_\Sigma(\neg C) = \mathbf{f_4}$, as desired.
- If $v_\Sigma(C) \in \{\mathbf{f_1}, \mathbf{f_3}\}$, then by IH, we obtain that $\Sigma \vdash \neg\Diamond\neg C$ and $\Sigma \nvdash C$ and $\Sigma \vdash \Diamond\neg\neg C$. From this, we immediately get $\Sigma \nvdash \neg\Diamond\neg\neg C$ and $\Sigma \vdash \neg C$ and $\Sigma \nvdash \Diamond\neg C$ and $\Sigma \nvdash \Diamond\neg\neg\neg C$. Then, by the definition of v_Σ, this means $v_\Sigma(\neg C) = \mathbf{t_4}$, as desired.

The other cases are similar and left to the reader.

Case 2. If $B = \Diamond C$, then we can deal with sixteen cases by splitting into the following two cases.

- If $v_\Sigma(C) \in \{\mathbf{T_i}, \mathbf{t_i}, \mathbf{f_i}, \mathbf{F_i}\}$ with $i \in \{1, 2\}$, then by IH, we obtain that $\Sigma \vdash \Diamond C$. By the definition of v_Σ, we obtain $v_\Sigma(\Diamond C) \in \mathcal{T}$, as desired.
- If $v_\Sigma(C) \in \{\mathbf{T_i}, \mathbf{t_i}, \mathbf{f_i}, \mathbf{F_i}\}$ with $i \in \{3, 4\}$, then by IH, we obtain that $\Sigma \nvdash \Diamond C$. By the definition of v_Σ, we obtain $v_\Sigma(\Diamond C) \notin \mathcal{T}$, i.e. $v_\Sigma(\Diamond C) \in \mathcal{F}$, as desired.

Case 3. If $B = C \to D$, then we have 256 different cases, which can be reduced to eighteen of which will prove four.

- If $v_\Sigma(D) \in \{\mathbf{T_1}, \mathbf{T_3}\}$, then by IH, we obtain that $\Sigma \vdash \neg\Diamond\neg D$ and $\Sigma \vdash D$ and $\Sigma \vdash \Diamond\neg\neg D$. From the first conjunct and (Ak$_f$4) we get $\Sigma \vdash \neg\Diamond\neg(C \to D)$. The second conjunct and (Ax1) gives us $\Sigma \vdash C \to D$ and by the third conjunct together with (Ak$_f$3) we have $\Sigma \vdash \Diamond\neg\neg(C \to D)$. Then, by the definition of v_Σ, this means $v_\Sigma(C \to D) \in \{\mathbf{T_1}, \mathbf{T_3}\}$, as desired.
- If $v_\Sigma(C) \in \{\mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}, \mathbf{F_4}\}$ and $v_\Sigma(D) \in \{\mathbf{t_1}, \mathbf{t_2}, \mathbf{t_3}, \mathbf{t_4}, \mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}, \mathbf{F_4}\}$, then by IH, we obtain that $\Sigma \nvdash \neg\Diamond\neg C$ and $\Sigma \nvdash C$ and $\Sigma \nvdash \neg\Diamond\neg D$. From the second conjunct together with (Ax1) and (Ax3) we infer $\Sigma \vdash C \to D$. And

the first conjunct together with ($\text{Ak}_\text{f}2$) gives us $\Sigma \vdash \Diamond\neg\neg(C \to D)$. Then, by the definition of v_Σ, this means $v_\Sigma(C \to D) \in \{\mathbf{T_1}, \mathbf{T_3}, \mathbf{t_1}, \mathbf{t_3}\}$, as desired.

- If $v_\Sigma(C) \in \{\mathbf{t_1}, \mathbf{t_2}, \mathbf{t_3}, \mathbf{t_4}\}$ and $v_\Sigma(D) \in \{\mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}, \mathbf{f_4}\}$, then by IH, we obtain that $\Sigma \nvdash \neg\Diamond\neg C$ and $\Sigma \vdash C$ and $\Sigma \vdash \neg\Diamond\neg D$ and $\Sigma \nvdash D$. The third conjunct together with ($\text{Ak}_\text{f}4$) gives us $\Sigma \vdash \neg\Diamond\neg(C \to D)$, while the first conjunct together with ($\text{Ak}_\text{f}2$) gives us $\Sigma \vdash \Diamond\neg\neg(C \to D)$. We also obtain $\Sigma \nvdash C \to D$ from the second and fourth conjunct together with (2) from Proposition 3.4. Then, by the definition of v_Σ, this means $v_\Sigma(C \to D) \in \{\mathbf{f_1}, \mathbf{f_3}\}$, as desired.
- If $v_\Sigma(C) \in \{\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}, \mathbf{T_4}\}$ and $v_\Sigma(D) \in \{\mathbf{f_2}, \mathbf{f_4}\}$, then by IH, we obtain that $\Sigma \vdash \neg\Diamond\neg C$ and $\Sigma \vdash C$ and $\Sigma \vdash \neg\Diamond\neg D$ and $\Sigma \nvdash D$ and $\Sigma \nvdash \Diamond\neg\neg D$. The third conjunct together ($\text{Ak}_\text{f}4$) gives us $\Sigma \vdash \neg\Diamond\neg(C \to D)$. From the second and forth conjunct together with (2) from Proposition 3.4 we obtain $\Sigma \nvdash (C \to D)$. And ($\text{Ak}_\text{f}1$), together with the first and the last conjuncts, gives us $\Sigma \nvdash \Diamond\neg\neg(C \to D)$. Then, by the definition of v_Σ, this means $v_\Sigma(C \to D) \in \{\mathbf{f_2}, \mathbf{f_4}\}$, as desired.

Truth-table for the conditional in Definition 2.7

$A \tilde{\to} B$	$\mathbf{T_1}$	$\mathbf{T_2}$	$\mathbf{T_3}$	$\mathbf{T_4}$	$\mathbf{t_1}$	$\mathbf{t_2}$	$\mathbf{t_3}$	$\mathbf{t_4}$	$\mathbf{f_1}$	$\mathbf{f_2}$	$\mathbf{f_3}$	$\mathbf{f_4}$	$\mathbf{F_1}$	$\mathbf{F_2}$	$\mathbf{F_3}$	$\mathbf{F_4}$
$\mathbf{T_1}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	t_{13}	t_{24}	t_{13}	t_{24}	f_{13}	f_{24}	f_{13}	f_{24}	$\mathcal{F}_{13}$	$\mathcal{F}_{24}$	$\mathcal{F}_{13}$	$\mathcal{F}_{24}$
$\mathbf{T_2}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	t_{13}	t_{24}	t_{13}	t_{24}	f_{13}	f_{24}	f_{13}	f_{24}	$\mathcal{F}_{13}$	$\mathcal{F}_{24}$	$\mathcal{F}_{13}$	$\mathcal{F}_{24}$
$\mathbf{T_3}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	t_{13}	t_{24}	t_{13}	t_{24}	f_{13}	f_{24}	f_{13}	f_{24}	$\mathcal{F}_{13}$	$\mathcal{F}_{24}$	$\mathcal{F}_{13}$	$\mathcal{F}_{24}$
$\mathbf{T_4}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	t_{13}	t_{24}	t_{13}	t_{24}	f_{13}	f_{24}	f_{13}	f_{24}	$\mathcal{F}_{13}$	$\mathcal{F}_{24}$	$\mathcal{F}_{13}$	$\mathcal{F}_{24}$
$\mathbf{t_1}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	f_{13}	f_{13}	f_{13}	f_{13}	$f_{13}, \mathcal{F}_{13}$	$f_{13}, \mathcal{F}_{13}$	$f_{13}, \mathcal{F}_{13}$	$f_{13}, \mathcal{F}_{13}$
$\mathbf{t_2}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	f_{13}	f_{13}	f_{13}	f_{13}	$f_{13}, \mathcal{F}_{13}$	$f_{13}, \mathcal{F}_{13}$	$f_{13}, \mathcal{F}_{13}$	$f_{13}, \mathcal{F}_{13}$
$\mathbf{t_3}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	f_{13}	f_{13}	f_{13}	f_{13}	$f_{13}, \mathcal{F}_{13}$	$f_{13}, \mathcal{F}_{13}$	$f_{13}, \mathcal{F}_{13}$	$f_{13}, \mathcal{F}_{13}$
$\mathbf{t_4}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	f_{13}	f_{13}	f_{13}	f_{13}	$f_{13}, \mathcal{F}_{13}$	$f_{13}, \mathcal{F}_{13}$	$f_{13}, \mathcal{F}_{13}$	$f_{13}, \mathcal{F}_{13}$
$\mathbf{f_1}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	t_{13}	t_{24}	t_{13}	t_{24}	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	t_{13}	t_{24}	t_{13}	t_{24}
$\mathbf{f_2}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	t_{13}	t_{24}	t_{13}	t_{24}	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	t_{13}	t_{24}	t_{13}	t_{24}
$\mathbf{f_3}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	t_{13}	t_{24}	t_{13}	t_{24}	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	t_{13}	t_{24}	t_{13}	t_{24}
$\mathbf{f_4}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	t_{13}	t_{24}	t_{13}	t_{24}	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	$\mathcal{T}_{13}$	$\mathcal{T}_{24}$	t_{13}	t_{24}	t_{13}	t_{24}
$\mathbf{F_1}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$
$\mathbf{F_2}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$
$\mathbf{F_3}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$
$\mathbf{F_4}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$	$\mathcal{T}_{13}, t_{13}$

References

[1] Avron, A. and A. Zamansky, *Non-deterministic semantics for logical systems*, in: *Handbook of Philosophical Logic, vol.16*, Springer, 2011 pp. 227–304.

[2] Blackburn, P., M. d. Rijke and Y. Venema, "Modal Logic," Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 2001.

[3] Coniglio, M. E., L. Fariñas del Cerro and N. Peron, *Finite non-deterministic semantics for some modal systems*, Journal of Applied Non-Classical Logics **25** (2015), pp. 20–45.

[4] Coniglio, M. E., L. Fariñas del Cerro and N. Peron, *Errata and Addenda to 'Finite non-deterministic semantics for some modal systems'*, Journal of Applied Non-Classical Logics **26** (2016), pp. 336–345.

[5] Coniglio, M. E., L. Fariñas del Cerro and N. Peron, *Modal logic with non-deterministic semantics: Part I—Propositional case*, Logic Journal of the IGPL **28** (2020), pp. 281–315.

[6] Copeland, B. J., *On When a Semantics is Not a Semantics: Some Reasons for Disliking the Routley-Meyer Semantics for Relevance Logic*, Journal of Philosophical Logic **8** (1979), pp. 399–413.

[7] French, R., *Notational variance and its variants*, Topoi **38** (2019), pp. 321–331.

[8] Hiż, H., *A warning about translating axioms*, The American Mathematical Monthly **65** (1958), pp. 613–614.

[9] Humberstone, L., *Yet another "choice of primitives" warning: Normal modal logics*, Logique et Analyse **185–188** (2004), pp. 395–407.

[10] Ivlev, Y. V., *A semantics for modal calculi*, Bulletin of the Section of Logic **17** (1988), pp. 114–121.

[11] Ivlev, Y. V., "Modal logic. (in Russian)," Moskva: Moskovskij Gosudarstvennyj Universitet, 1991.

[12] Kearns, J., *Modal Semantics without Possible Worlds*, Journal of Symbolic Logic **46** (1981), pp. 77–86.

[13] Kearns, J., *Leśniewski's strategy and modal logic*, Notre Dame Journal of Formal Logic **30** (1989), pp. 77–86.

[14] Makinson, D., *A warning about the choice of primitive operators in modal logic*, Journal of Philosophical Logic **2** (1973), pp. 193–196.

[15] Marcos, J., *Modality and paraconsistency*, in: M. Bilková and L. Běhounek, editors, *The Logica Yearbook 2004*, Filosofia, 2005 pp. 213–222.

[16] Omori, H. and D. Skurt, *More modal semantics without possible worlds*, IfCoLog Journal of Logics and their Applications **3** (2016), pp. 815–846.

[17] Scroggs, S. J., *Extensions of the Lewis system S5*, Journal of Symbolic Logic **16** (1951), pp. 112–120.

[18] Shramko, Y. and H. Wansing, *Some useful 16-valued logics: How a computer network should think*, Journal of Philosophical Logic **34** (2005), pp. 121–153.

[19] Shramko, Y. and H. Wansing, "Truth and Falsehood - An Inquiry into Generalized Logical Values," Trends in Logic **36**, Springer Netherlands, 2012.

[20] Tarski, A. and J. Łukasiewicz, *Investigations into the sentential calculus*, in: *Logic, Semantics, Metamathematics; Collected Papers of A. Tarski*, Oxford University Press, 1956 pp. 38–59.

[21] Thomason, R., *Independence of the Dual Axiom in Modal* **K** *with Primitive* $\Diamond$, Notre Dame Journal of Formal Logic **59** (2018), pp. 381–385.

[22] Thomason, S. K., *Semantic analysis of tense logics*, Journal of Symbolic Logic **37** (1972), pp. 150–158.

[23] Wen, X., *Some Common Mistakes in the Teaching and Textbooks of Modal Logic (in Chinese)*, Studies in Logic **11** (2018), pp. 69–86, English translation is available at: `https://arxiv.org/abs/2005.10137`.