Bi-Intuitionistic Logics: a New Instance of an Old Problem

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Abstract

As anyone who reads the literature on bi-intuitionistic logic will know, the numerous papers by Cecylia Rauszer are foundational but confusing. For example: these papers claim and retract various versions of the deduction theorem for bi-intuitionistic logic; they erroneously claim that the calculus is complete with respect to rooted canonical models; and they erroneously claim the admissibility of cut in her sequent calculus for this logic. Worse, authors such as Crolard, have based some of their own foundational work on these confused and confusing results and proofs.
We trace this confusion to the axiomatic formalism of RBiInt in which Rauszer first characterized bi-intuitionistic logic and show that, as in modal logic, RBiInt can be interpreted as two different consequence relations. We remove this ambiguity by using generalized Hilbert calculi, which are tailored to capture consequence relations.
We show that RBiInt leads to two logics, wBIL and sBIL, with different extensional and meta-level properties, and that they are, respectively, sound and strongly complete with respect to the Kripkean local and global semantic consequence relations of bi-intuitionistic logic. Finally, we explain where they were conflated by Rauszer.

Keywords: Bi-Intuitionistic Logic, Axiomatic Proof Theory, Consequence Relations, Deduction Theorems, Kripke Semantics.

1 Introduction: Confusions

Rauszer's Bi-Intuitionistic logic (RBiInt), introduced in 1974 via an axiomatic calculus [17], is a conservative extension of intuitionistic propositional logic. It adds an extra binary operator $\prec$, dual to the intuitionistic arrow and variously called *exclusion*, *subtraction*, or *co-implication*, and a unary *weak* negation operator $\sim$ definable from $\prec$. In an interdependent series of articles [16,18,19,20,21,22,23], Rauszer studied the algebraic, axiomatic and Kripke-style aspects of this logic. Alas, reviewing the literature on RBiInt can be quite confusing, because, in many places, the status of theorems is unclear if not puzzling. An account of this confusion can be given by three elements.

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First, as is well-known, the usual deduction theorem is: $\Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash \varphi \to \psi$. However the "deduction theorem" is claimed under the following various forms in chronological order: (1) $\Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash \neg{\sim}...\neg{\sim}\varphi \to \psi$ [17]; (2) the usual version above [18]; an explicit retraction of (2) and replacement by (3) $\Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash \neg{\sim}\varphi \to \psi$ [21]; (4) a return to (1) without retracting (3) [23]. Crolard [3] claims that yet another form of the deduction theorem fails to hold.

Second, the Pinto-Uustalu counterexample [14] not only breaks the admissibility of cut in Rauszer's sequent calculus [16] for RBiInt, but also casts doubts on Crolard's work on a formulas-as-types interpretation for RBiInt because of his claim that *"as a by-product of the previous properties [proved by Crolard], we obtain a new proof of this result [Rauszer's cut admissibility]"* [4, p.3].

Third, Rauszer's [20] strong completeness of RBiInt w.r.t. rooted canonical models contradicts Crolard's [3] result that it is not complete for this class.

All this confusion arises from a fundamental problem in the axiomatic proof theory of RBiInt: traditional Hilbert calculi are not designed to treat logics as consequence relations. They lead to an ambiguous notion of derivation from assumptions that can cause us to conflate distinct logics. For example, modal logic as a consequence relation splits into a *strong* and a *weak* version, depending on how the *necessitation rule* is interpreted. Conflating these logics leads to great confusion, notably regarding the deduction theorem [11]. A similar phenomenon, as yet undetected, occurs in RBiInt where traditional Hilbert calculi cannot adequately separate two interpretations of a bi-intuitionistic rule called DN (an analogue of the necessitation rule from modal logic).

To pinpoint the confusion, we generalize traditional Hilbert calculi to treat consequence relations rather than just theoremhood. Then, the rules, such as necessitation or DN, are expressed in a way that prevents ambiguities about their shape. We use such calculi to explain and fix the fundamental problem of the axiomatic proof theory of RBiInt. Specifically, we give two generalized Hilbert calculi, wBIC and sBIC, for bi-intuitionistic logic that differ only in the shape of the DN rule. Unsurprisingly, these systems capture two distinct logics **wBIL** and **sBIL**, which have been conflated in parts of the literature.

Finally, the logics **wBIL** and **sBIL** are shown, respectively, to be sound and strongly complete w.r.t. the Kripkean *local* and *global* semantic consequence relations for bi-intuitionistic logic, mimicking similar results in modal logic via a canonical model construction while using techniques of Sano and Stell [25].

Section 2 contains general definitions of logics as consequence relations and generalized Hilbert calculi. Section 3 contains the problems caused by traditional Hilbert calculi in modal logics, and how generalized Hilbert calculi solve them. Rauszer's traditional Hilbert calculus is in Section 4. Section 5 contains the two generalized Hilbert calculi obtained from Rauszer's axiomatization. In Section 6, we show they define two extensionally distinct logics. Section 7 contains significant theorems distinguishing these logics. Section 8 contains our completeness proofs. In Section 9, we use these results to prove pending claims from Sections 5 and 7. In Section 10, we use the distinctions between our two bi-intuitionistic logics to expose the flaws in Rauszer's results.

2 Preliminaries

In this section we provide the general definitions required to both understand confusions arising in both modal and bi-intuitionistic logic, and avoid them.

We define logics as conventional consequence relations [6,12] where uniform substitution, formal language $\mathcal{L}$ and the set $\mathcal{F}orm_{\mathcal{L}}$ of all formulae of $\mathcal{L}$ is as usual. We use $\varphi, \psi, \chi, ...$ for formulae and $\Gamma, \Delta, ...$ for sets of formulae.

Definition 2.1 Let $\mathcal{L}$ be a formal language. A logic in $\mathcal{L}$ is a set $L \subseteq \{(\Gamma, \varphi) \mid \Gamma \cup \{\varphi\} \subseteq \mathcal{F}orm_{\mathcal{L}}\}$ that satisfies the following properties:

Identity: if $\varphi \in \Gamma$, then $(\Gamma, \varphi) \in L$;

Monotonicity: if $(\Gamma, \varphi) \in L$ and $\Gamma \subseteq \Gamma'$, then $(\Gamma', \varphi) \in L$;

Compositionality: if $(\Gamma, \varphi) \in L$ and $(\Delta, \gamma) \in L$ for all $\gamma \in \Gamma$, then $(\Delta, \varphi) \in L$;

Structurality: if $(\Gamma, \varphi) \in L$, then $(\Gamma^\sigma, \varphi^\sigma) \in L$ for uniform substitution σ.

A logic L is *finitary* if $(\Gamma, \varphi) \in L$ implies there is a finite $\Gamma' \subseteq \Gamma$ with $(\Gamma', \varphi) \in L$.

Thus, technically, a logic is not just a set of theorems but is a consequence relation containing pairs (Γ, φ). Wójcicki [27, pp.xii-xiii, pp.43-51] discusses some interesting aspects of this notion. We then formalize axiomatic systems in a way that generalizes and disambiguates traditional Hilbert calculi. In what follows the notions of formula schema and schema instance are as usual. The letters $A, B, C...$ refer to schemata and $X, Y, Z, ...$ to sets of schemata. We call the axiomatic systems obtained *generalized Hilbert calculi.*

Definition 2.2 Let $\mathcal{L}$ be a language. An *axiom* is a formula schema of $\mathcal{L}$. If $\mathcal{A}$ is a set of axioms, we define $\mathcal{A}^I$ to be the set of instances of axioms of $\mathcal{A}$. An n-ary *rule* $R = (\mathbb{P}, \mathbb{C})$ is a pair where $\mathbb{P} = \{X_1 \vdash B_1, ..., X_n \vdash B_n\}$ is a set of n premises and $\mathbb{C} = (X_{n+1} \vdash B_{n+1})$ is the conclusion, and $\bigcup_{i=1}^{n+1} X_i \cup \{B_i\}$ is a set of schemata of formulae. If R is a rule then we define R^I to be the set of instances of R. A generalized Hilbert calculus in $\mathcal{L}$ is a pair $S = (\mathcal{A}, \mathcal{R})$.

To let a generalized Hilbert calculus define a binary relation we must say which statements of the form $\Gamma \vdash \varphi$ follow from this calculus. To do so, we need to define the notion of derivation in a generalized Hilbert calculus:

Definition 2.3 Let $\mathcal{L}$ be a language. Let $\Gamma \cup \{\varphi\} \in Form_{\mathcal{L}}$ and $S = (\mathcal{A}, \mathcal{R})$ a generalized Hilbert calculus in $\mathcal{L}$. A derivation in S is a tree of expressions, defined inductively as follows:

(Ax): if $\varphi \in \mathcal{A}^I$ then the following is a derivation: $\dfrac{}{\Gamma \vdash \varphi}\ Ax$

(El): if $\varphi \in \Gamma$ then the following is a derivation: $\dfrac{}{\Gamma \vdash \varphi}\ El$

(R): if $\pi_1, \pi_2, ..., \pi_k$ are derivations with respectively $\Gamma_1 \vdash \varphi_1, ..., \Gamma_k \vdash \varphi_k$ as roots and $(\{\Gamma_1 \vdash \varphi_1, ..., \Gamma_k \vdash \varphi_k\}, \Gamma \vdash \varphi) \in R^I$ for some $R \in \mathcal{R}$, then the following is a derivation:

$$\dfrac{\pi_1 \quad ... \quad \pi_k}{\Gamma \vdash \varphi}\ R$$

A branch, its length and the length $l(\pi)$ of a derivation π are defined as usual. If there is a derivation in S with $\Gamma \vdash \varphi$ as root, we write $\Gamma \vdash_S \varphi$.

Note that a generalized Hilbert calculus might not define a logic: the relation defined by a system with the lone rule $R = (\emptyset, A \vdash B)$ fails Monotonicity.

3 Theorems and Consequences in Classical Modal Logic

As an example, generalized Hilbert calculi clearly demarcate the existence of two modal logics based on the usual axiomatization $\mathcal{A}_{\mathbf{K}}$ of the basic modal logic **K**. In that setting, the *modus ponens* rule MP is formalized as:

$$\frac{X \vdash A \quad X \vdash A \to B}{X \vdash B}\ {\scriptstyle MP}$$

The *Necessitation* rule, often written as in the middle, can be interpreted either as a <u>w</u>eak or <u>s</u>trong rule as shown at left and right.

$$\frac{\emptyset \vdash A}{X \vdash \Box A}\ {\scriptstyle Nec_w} \qquad\qquad \frac{A}{\Box A}\ {\scriptstyle Nec} \qquad\qquad \frac{X \vdash A}{X \vdash \Box A}\ {\scriptstyle Nec_s}$$

The calculi wKC $= (\mathcal{A}_{\mathbf{K}}, \{MP, Nec_w\})$ and sKC $= (\mathcal{A}_{\mathbf{K}}, \{MP, Nec_s\})$ respectively define the (*distinct*) logics **wK** and **sK**, corresponding to the extensionally different *local* and *global* Kripkean semantic consequence relations [12].

The most obvious example of their difference, as consequence relations, is that we have $p \vdash_{\mathrm{sKC}} \Box p$ but $p \nvdash_{\mathrm{wKC}} \Box p$. Then, the long-standing debate [11] about the modal deduction theorem is resolved immediately via two simple facts: (1) $p \vdash_{\mathrm{sKC}} \Box p$ but $\nvdash_{\mathrm{sKC}} p \to \Box p$; (2) $p \vdash_{\mathrm{wKC}} \Box p$ iff $\vdash_{\mathrm{wKC}} p \to \Box p$.

Not only does this example show that the two rules added to the same axiomatization do not capture the same logics, as consequence relations, but it also gives sufficient tools to show that these logics differ on their metaproperties. In fact, this partly justifies the fact that the deduction theorem doesn't hold for **sK**, while it is proven to hold for **wK**.

That is, traditional Hilbert calculi allow us to easily confuse the logics **wK** and **sK**. To capture both of them in a traditional Hilbert setting, one has to provide debatable modifications on the notion of derivation. In fact, to capture **sK** one defines the notion of derivation from assumptions as follows [2]:

Definition 3.1 A derivation of φ from assumptions Γ is a list l of formulae ending with φ such that each formula in l is an instance of an axiom of $\mathcal{A}_{\mathbf{K}}$, a member of Γ, or follows via MP or Nec from formulae appearing earlier in l.

While this definition is natural and unproblematic, the notion of derivation from assumptions has to be bent to capture **wK**:

Definition 3.2 A derivation of φ from assumptions Γ is a list of formulae ending with φ, and such that every formula in the list is an instance of an axiom, a member of Γ, follows from formulae appearing before it in the list by MP or *follows from a derivable formula by* Nec.

First, this definition relies on the notion of derivability which really should just be a special case of derivation from assumptions. Second, as it involves the derivability of a formula in the application of Nec, to determine if a list of formulae is a derivation from assumptions or not it is not sufficient to check the list of formulae itself. In other words, the notion defined here is not local as the application of Nec is conditioned on the existence of another derivation. These features bring a lot of confusion on the nature of derivations from assumptions.

A common way to avoid these contortions is to define the notion of derivation from assumptions from the notion of derivation [1,15]:

Definition 3.3 A derivation of φ from assumptions Γ is a derivation of the formula $(\gamma_0 \wedge ... \wedge \gamma_n) \to \varphi$ for some $n \in \mathbb{N}$ and $\gamma_i \in \Gamma$ for $0 \leq i \leq n$.

Here, some other criticisms can be given. Mainly, it is the striking lack of generality of this definition that we address. More precisely, this definition is not general as there are four types of logics that it cannot capture. First, logics without a conjunction, such as implicational ticket entailment, cannot be captured. Second, the same remark can be made of logics devoid of implication, such as positive modal logic and geometric logic. Third, logics that are not compact are ruled out: it is in their nature to be unable, in some circumstances, to reduce an infinite set of assumptions to a finite one, while this is forced here by the presence of $\gamma_0, ..., \gamma_n$. Finally, no logic for which the deduction theorem fails can be characterized via this definition, as this theorem is built in here.

Generalized Hilbert calculi avoid these issues while easily capturing the logic **wK** by interpreting the necessitation rule as Nec_w. Of course, all of this is well-known for modal logic. We next use generalized Hilbert calculi to show that RBiInt is the victim of a similar confusion: whence our title.

4 Rauszer's Hilbert Calculus for Bi-Intuitionistic Logic

Before showing how bi-intuitionistic logic is captured via generalized Hilbert calculi, we recall Rauszer's traditional Hilbert calculus RBiInt from 1974 [20].

As mentioned above, RBiInt is expressed in the language of intuitionistic logic extended with two operators, i.e. $\prec$ and $\sim$. More formally:

Definition 4.1 Let p, q, r range over a countable set $Prop$ of propositional atoms and let $Log_{BI} = \{\wedge, \vee, \to, \neg, \prec, \sim\}$ be the set of bi-intuitionistic logical connectives. This pair forms the the language $\mathcal{L}_{BI} := (Log_{BI}, Prop)$ of bi-intuitionistic logic. The formulae $Form_{BI}$ of $\mathcal{L}_{BI}$ are defined as follows:

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \neg\varphi \mid \varphi \prec \varphi \mid \sim\varphi$$

For convenience, we define $\top := p \to p$ and $\bot := p \prec p$ for some fixed atomic formula p. The added operators are meant to be the duals of, respectively, $\to$ and $\neg$. The formula $\varphi \prec \psi$ is usually read as "φ excludes ψ". The formula $\sim \varphi := \top \prec \varphi$, defined dually to $\neg\varphi := \varphi \to \bot$, is usually called "weak negation". Rauszer's traditional Hilbert calculus RBiInt is defined next [17]:

Definition 4.2 RBiInt consists of the axioms $\mathcal{A}_{BI}$ and rules $\mathcal{R}_{BI}$ below:

RA_1	$(A \to B) \to ((B \to C) \to (A \to C))$	RA_{10}	$(A \to B) \to (\neg B \to \neg A)$
RA_2	$A \to (A \vee B)$	RA_{11}	$A \to (B \vee (A \prec B))$
RA_3	$B \to (A \vee B)$	RA_{12}	$(A \prec B) \to \sim(A \to B)$
RA_4	$(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$		
RA_5	$(A \wedge B) \to A$	RA_{14}	$\neg(A \prec B) \to (A \to B)$
RA_6	$(A \wedge B) \to B$	RA_{15}	$(A \to \bot) \to \neg A$
RA_7	$(A \to B) \to ((A \to C) \to (A \to (B \wedge C)))$	RA_{16}	$\neg A \to (A \to \bot)$
RA_8	$(A \to (B \to C)) \to ((A \wedge B) \to C)$	RA_{17}	$(\top \prec A) \to \sim A$
RA_9	$((A \wedge B) \to C) \to (A \to (B \to C))$	RA_{18}	$\sim A \to (\top \prec A)$

$$RA_{13}\ ((A \prec B) \prec C) \to (A \prec (B \vee C)) \qquad \frac{A \quad A \to B}{B}\ {\scriptstyle MP} \qquad \frac{A}{\neg \sim A}\ {\scriptstyle DN}$$

Next, we show that the *Double Negation* rule DN can be interpreted in the context of generalized Hilbert calculi in two main ways, giving different logics.

5 Bi-Intuitionistic Logic As a Consequence Relation

As in the modal case, the traditional Hilbert calculus hides a distinction in the shape of rules. To be more precise, it overlooks the multiple interpretations of DN that are clearly expressed in a generalized Hilbert calculus:

$$\frac{\emptyset \vdash A}{X \vdash \neg\sim A}\ {\scriptstyle DN_w} \qquad\qquad \frac{X \vdash A}{X \vdash \neg\sim A}\ {\scriptstyle DN_s}$$

As we shall see, not only are these rules formally different, but they also have significantly different strength, implying a difference in the consequence relations they define and hence a difference in their logics. To see the difference between the two logics, erroneously identified in Rauszer's work, that emerge from the set of axioms $\mathcal{A}_{BI}$, we define the following generalized Hilbert calculi.

Definition 5.1 We define the generalized Hilbert calculi wBIC $= (\mathcal{A}_{BI}, \mathcal{R}_w)$ and sBIC $= (\mathcal{A}_{BI}, \mathcal{R}_s)$, where $\mathcal{R}_w = \{MP, DN_w\}$ and $\mathcal{R}_s = \{MP, DN_s\}$. We abbreviate $\Gamma \vdash_{wBIC} \varphi$ by $\Gamma \vdash_w \varphi$ and let **wBIL** $= \{(\Gamma, \varphi) \mid \Gamma \vdash_w \varphi\}$ be the consequence relation characterized by wBIC. Similarly we abbreviate $\Gamma \vdash_{sBIC} \varphi$ by $\Gamma \vdash_s \varphi$, and define **sBIL** $= \{(\Gamma, \varphi) \mid \Gamma \vdash_s \varphi\}$.

As there is no guarantee that generalized Hilbert calculi define logics, to assert that **sBIL** and **wBIL** are logics we must show they satisfy Definition 2.1. The single rule derivation of $\Gamma \vdash \varphi$ via (El) shows that **Identity** is satisfied both in **sBIL** and **wBIL**. The other properties need to be proved.

Lemma 5.2 *The following holds for $i \in \{w, s\}$:*

Monotonicity: *if $\Gamma \subseteq \Gamma'$ and $\Gamma \vdash_i \varphi$ then $\Gamma' \vdash_i \varphi$.*

Compositionality: *if $\Gamma \vdash_i \varphi$ and $\Delta \vdash_i \gamma$ for all $\gamma \in \Gamma$, then $\Delta \vdash_i \varphi$*

Structurality: *if $\Gamma \vdash_i \varphi$ then $\Gamma^\sigma \vdash_i \varphi^\sigma$.*

Proof. See the Appendix. □

We are now in position to claim that **sBIL** and **wBIL** are both logics. Furthermore, we can add that they are finitary logics:

Lemma 5.3 *For $i \in \{s, w\}$, if $\Gamma \vdash_i \varphi$, then $\Gamma' \vdash_i \varphi$ for some finite $\Gamma' \subseteq \Gamma$.*

That **sBIL** and **wBIL** are (finitary) logics is all well and good, but we require further work to show that they are *different* logics, as explained next.

6 Extensional Interactions

To prove our claim that **sBIL** and **wBIL** are two logics that were erroneously conflated in the literature we first show they differ on an extensional level.

Claim 6.1 *For $p \in Prop$, $p \vdash_s \neg{\sim}p$ and $p \nvdash_w \neg{\sim}p$.*

While it is clear that $p \vdash_s \neg{\sim}p$ holds because DN_s can be applied on $p \vdash p$, we need a semantic argument, that we provide later, to prove that $p \nvdash_w \neg{\sim}p$. By accepting this result for now, we can see that the two consequence relations **sBIL** and **wBIL** are extensionally different. However the two consequence relations are closely related. In fact, **sBIL** is an extension of **wBIL**:

Theorem 6.2 *If $\Gamma \vdash_w \varphi$ then $\Gamma \vdash_s \varphi$.*

Moreover, they coincide on their sets of *theorems* (derivable from $\emptyset$):

Theorem 6.3 *$\emptyset \vdash_s \varphi$ if and only if $\emptyset \vdash_w \varphi$.*

Traditionally, Theorem 6.3 is an argument against our distinction between **sBIL** and **wBIL** as it identifies the two logics on their sets of theorems. However, as mentioned previously, they are different consequence relations. Given Claim 6.1, **sBIL** and **wBIL** are thus different logics.

7 Deduction and Dual-Deduction Theorems

We proceed to show that **sBIL** and **wBIL** are distinct on a meta-level by proving that both the deduction theorem and its dual hold for **wBIL**, while none hold for **sBIL**. To express these statements we use notions from Sano and Stell [25]. They can be interpreted as an extension of the notion of a logic as a consequence relation of the form (Γ, φ) to the more general form (Γ, Δ).

Definition 7.1 Let $i \in \{w, s\}$ and $\bigvee \Delta$ be the disjunction of all the members of Δ. We define the following:

(i) $\vdash_i [\Gamma \mid \Delta]$ if $\Gamma \vdash_i \bigvee \Delta'$ for some finite $\Delta' \subseteq \Delta$;

(ii) $\nvdash_i [\Gamma \mid \Delta]$ if it is not the case that $\vdash_i [\Gamma \mid \Delta]$;

(iii) $\bigvee \Delta := \bot$ if $\Delta = \emptyset$;

(iv) $[\Gamma \mid \Delta]$ is complete if $\Gamma \cup \Delta = \mathcal{F}orm_{BI}$.

Pairs of the form $[\Gamma \mid \Delta]$ bring a symmetry, witnessed by the presence of potentially infinite sets of formulae on both sides of the vertical bar, which is not present in expressions such as $\Gamma \vdash \varphi$. Conceptually, this symmetry and the presence of a non-orientated separation symbol $\mid$ suggests a bidirectional reading of a pair $[\Gamma \mid \Delta]$. From left to right such a pair should be read as a *deduction*, while from right to left it should be read as a *refutation*. This interpretation help us understand the duality between $\to$ and $\prec$.

We first require a preliminary result central to the two logics.

Proposition 7.2 *For $i \in \{w, s\}$:*

$$\vdash_i [\emptyset \mid (\varphi \prec \psi) \to \chi] \quad \textit{iff} \quad \vdash_i [\emptyset \mid \varphi \to (\psi \vee \chi)]$$

We have not given Rauszer's [17] algebraic semantics for RBiInt, but Proposition 7.2 is an object language analogue of the dual residuation property below:

$$\frac{a \leq b \vee c}{\overline{a \prec b \leq c}}$$

The deduction theorem is the first theorem to separate the two logics.

Theorem 7.3 (Deduction Theorem) **wBIL** *enjoys the deduction theorem:*

$$\vdash_w [\Gamma, \varphi \mid \psi] \quad \textit{iff} \quad \vdash_w [\Gamma \mid \varphi \to \psi]$$

Next, we give a counter-example for the deduction theorem for **sBIL**.

Proposition 7.4 *We have that $\vdash_s [p \mid \neg\sim p]$ but $\nvdash_s [\emptyset \mid p \to \neg\sim p]$.*

Proof. We prove the first conjunct and postpone the proof of the second to later. Obviously we have $p \vdash_s p$. So we can apply the rule DN_s to obtain $p \vdash_s \neg\sim p$, hence $\vdash_s [p \mid \neg\sim p]$. □

We leave the following claim as pending:

Claim 7.5 *We have that $\nvdash_s [\emptyset \mid p \to \neg\sim p]$.*

This situation is very similar to the modal case: it is well-known that **wK** satisfies the deduction theorem while **sK** does not. However, a variant of this theorem does hold for **sK**: $\Gamma, \varphi \vdash_s \psi$ *iff there exists a $n \in \mathbb{N}$ such that* $\Gamma \vdash_s (\varphi \wedge \Box\varphi \wedge ... \wedge \Box^n\varphi) \to \psi$ [2, p.85]. A similar variant of the deduction theorem holds for **sBIL**, but we first need some notation to express it.

Definition 7.6 We define:

(i) for $n \in \mathbb{N}$, let $(\neg\sim)^0\varphi := \varphi$ and let $(\neg\sim)^{(n+1)}\varphi := \neg\sim(\neg\sim)^n\varphi$;

(ii) $(\neg\sim)^n\Gamma = \{(\neg\sim)^n\gamma \mid \gamma \in \Gamma\}$;

(iii) $(\neg\sim)^\omega\Gamma = \bigcup_{n\in\mathbb{N}} (\neg\sim)^n\Gamma$.

The variant of the deduction theorem below uses the pattern $\neg\sim$ as the modal variant uses $\Box$. But it suffices to replace φ by just $(\neg\sim)^n\varphi$, without the conjunction of all $(\neg\sim)^i\varphi$ for $i \leq n$, as $\neg\sim$ is a **T** modality satisfying $\neg\sim\varphi \to \varphi$. One reviewer noted that Reyes and Zolfaghari [24] show how to interpret this combination as a kind of non-idempotent interior operation on subgraphs.

Theorem 7.7 (Double-Negated Deduction Theorem)

$$\vdash_s [\Gamma, \varphi \mid \psi] \quad \textit{iff} \quad \exists n \in \mathbb{N} \; s.t. \; \vdash_s [\Gamma \mid (\neg\sim)^n\varphi \to \psi]$$

Theorem 7.8 (Dual Deduction Theorem) *The following holds:*

$$\vdash_w [\varphi \mid \psi, \Delta] \qquad \textit{iff} \qquad \vdash_w [\varphi \prec \psi \mid \Delta].$$

Proof. Assume that $\vdash_w [\varphi \mid \psi, \Delta]$. By definition we get $\varphi \vdash_w \psi \vee \bigvee \Delta'$ where $\Delta' \subseteq \Delta$ is finite. Using Theorem 7.3 we get $\emptyset \vdash_w \varphi \to (\psi \vee \bigvee \Delta')$. We obtain $\emptyset \vdash_w (\varphi \prec \psi) \to \bigvee \Delta'$ by Proposition 7.2. By Theorem 7.3 again, we obtain $\varphi \prec \psi \vdash_w \bigvee \Delta'$. By definition we get $\vdash_w [\varphi \prec \psi \mid \Delta]$. Note that all the steps used here are based on equivalences. □

Before demonstrating that **sBIL** fails the dual deduction theorem, we remark on the previous theorem. Pairs $[\Gamma \mid \Delta]$ express the duality between $\to$ and $\prec$ on the syntactic level in **wBIL** by showing that $\prec$ plays the same role as $\to$ on the left-hand side of $|$: it internalizes in the object language the relation expressed by our pairs. Just as $\to$ internalizes the deduction relation of expressions such as $\Gamma \vdash \varphi$, dually $\prec$ internalizes the *refutation* relation of expressions such as $\Delta \dashv \varphi$, read "Δ refutes φ" and formalized here as $[\varphi \mid \Delta]$. This interpretation relies on the aforementioned reading of our pairs, from right to left, to express refutations. Fortunately, as we shall show in a separate paper, we can support this interpretation by the fact that **wBIL** can simulate the propositional fragment of Rauszer's refutation system [20, pp.62-63].

The following witnesses the failure of the dual deduction theorem for **sBIL**.

Proposition 7.9 $\vdash_s [p \prec q \mid \neg{\sim}{\sim} q]$ *while* $\nvdash_s [p \mid q, \neg{\sim}{\sim} q]$.

Proof. First, let us prove that $\vdash_s [p \prec q \mid \neg{\sim}{\sim} q]$. By definition, we need to show that $p \prec q \vdash_s \neg {\sim}{\sim} q$. We have that $\emptyset \vdash_w q \vee \sim q$, hence $\emptyset \vdash_w p \to (q \vee \sim q)$. By Proposition 7.2 we obtain $\emptyset \vdash_w (p \prec q) \to {\sim} q$. In turn, by Theorem 7.7 we get $p \prec q \vdash_w \sim q$. Then, by Theorem 6.2 we get that $p \prec q \vdash_s \sim q$. Finally, we can apply the rule DN_s to obtain $p \prec q \vdash_s \neg{\sim}{\sim} q$, hence $\vdash_s [p \prec q \mid \neg{\sim}{\sim} q]$. We leave the following claim as pending:

Claim 7.10 $\nvdash_s [p \mid q, \neg{\sim}{\sim} q]$.

□

While a variant of the deduction theorem exists for **sBIL**, the form or the existence of a variant to the dual deduction theorem is still a mystery to us. For the interested reader: while the deduction theorem fails for **sBIL** because of the rule DN_s, the dual deduction theorem fails for this logic because of the rule MP. It appears that if a variant of the dual deduction theorem exists for **sBIL**, then it must use a "patch" inspired by the structure of MP, as done in the double-negated deduction theorem with DN_s.

On top of the extensional difference between **wBIL** and **sBIL**, the deduction and dual deduction theorems expose their meta-difference. But both differences rely on claims that are still pending. The next section builds on Rauszer's Kripke semantics to resolve these claims.

8 Weak is Local and Strong is Global

wBIL and **sBIL**, proof-theoretically characterized via the generalized Hilbert calculi $wBIC$ and $sBIC$, can be captured model-theoretically in a Kripke semantics using well-known notions of semantic consequence: a *local* and a *global* one. This section is devoted to proving these claims.

First we need to define the Kripke semantics [23].

Definition 8.1 A BI-Kripke model $\mathcal{M}$ is a tuple $(W, \leq, I)$, where $(W, \leq)$ is a poset and $I : Prop \to \mathcal{P}(W)$ is an interpretation function obeying persistence: for every $v, w \in W$ with $w \leq v$ and $p \in Prop$, if $w \in I(p)$ then $v \in I(p)$.

The forcing relation of intuitionistic logic is extended to $\prec$ and $\sim$:

Definition 8.2 Given a BI-Kripke model $\mathcal{M} = (W, \leq, I)$, we extend the usual intuitionistic forcing relation between a point $w \in W$ and a formula as follows:

$$\begin{array}{lll} \mathcal{M}, w \Vdash \varphi \prec \psi & \text{iff} & \text{there exists a } v \text{ s.t. } v \leq w,\ \mathcal{M}, v \Vdash \varphi \text{ and } M, v \nVdash \psi \\ \mathcal{M}, w \Vdash \sim\varphi & \text{iff} & \text{there exists a } v \text{ s.t. } v \leq w,\ \mathcal{M}, v \nVdash \varphi \end{array}$$

Let $\Gamma \subseteq \mathcal{F}orm_{BI}$. We write $\mathcal{M}, w \Vdash \Gamma$ if for every $\gamma \in \Gamma$ we have $\mathcal{M}, w \Vdash \gamma$. If $\mathcal{M}, w \Vdash \Gamma$ we say that w is a Γ-point. We write $\mathcal{M} \Vdash \Gamma$ if for every point $w \in W$, $\mathcal{M}, w \Vdash \Gamma$. If $\mathcal{M} \Vdash \Gamma$ we say that $\mathcal{M}$ is a Γ-model.

The main feature of the Kripke semantics for intuitionistic logic is arguably persistence. This property, which we use later, is preserved here:

Lemma 8.3 (Persistence) *Let $\mathcal{M} = (W, \leq, I)$ be a BI-Kripke model and $w \in W$. For all $v \in W$ s.t. $w \leq v$ we have that if $\mathcal{M}, w \Vdash \varphi$ then $\mathcal{M}, v \Vdash \varphi$.*

We are now in position to define the two following notions of semantic consequence in the above-defined Kripke semantics:

Definition 8.4 The local and global consequence relations are as below:

$$\begin{array}{lll} \Gamma \models_l \Delta & \text{iff} & \forall \mathcal{M}. \forall w. (\mathcal{M}, w \Vdash \Gamma \Rightarrow \exists \delta \in \Delta. \mathcal{M}, w \Vdash \delta) \\ \Gamma \models_g \Delta & \text{iff} & \forall \mathcal{M}. (\mathcal{M} \Vdash \Gamma \Rightarrow \forall w \in W. \exists \delta \in \Delta. \mathcal{M}, w \Vdash \delta). \end{array}$$

While the two notions are not generally equivalent in modal logic, they are equivalent in intuitionistic (not bi-intuitionistic) logic. It is easy to see that the local implies the global in full generality. The converse holds for intuitionistic logic for two reasons. First, persistence plays an important role: if a formula is true at a point then it is true at all the successors (the upcone) of that point. Second and more crucially, in an intuitionistic Kripke model, the upcone of a point is bisimilar [2, p.54][13, p.8] in that point with the model itself.

Nonetheless, in the semantics just defined it is not the case that $\Gamma \models_g \Delta$ implies $\Gamma \models_l \Delta$. This can easily be shown by the fact that $p \models_g \neg \sim p$ while $p \not\models_l \neg \sim p$. This fact will help us finally establish the extensional difference between **wBIL** and **sBIL** by proving that local semantic consequence corresponds to **wBIL** and global semantic consequence corresponds to **sBIL**.

We use canonical models on complete pairs $[\Gamma \mid \Delta]$ from Sano and Stell [25]:

Definition 8.5 The canonical model $\mathcal{M}^c = (W^c, \leq^c, I^c)$ is defined in the following way:

(i) $W^c = \{[\Gamma \mid \Delta] : [\Gamma \mid \Delta] \text{ is complete and } \nvdash_w [\Gamma \mid \Delta]\}$;

(ii) $[\Gamma_1 \mid \Delta_1] \leq^c [\Gamma_2 \mid \Delta_2]$ iff $\Gamma_1 \subseteq \Gamma_2$;

(iii) $I^c(p) = \{[\Gamma \mid \Delta] \in W^c : p \in \Gamma\}$.

These pairs are built from unprovable pairs using a bi-intuitionistic version of the Lindenbaum Lemma:

Lemma 8.6 (Lindenbaum Lemma) *If* $\nvdash_w [\Gamma \mid \Delta]$ *then there exist* $\Gamma' \supseteq \Gamma$ *and* $\Delta' \supseteq \Delta$ *such that* $[\Gamma' \mid \Delta']$ *is complete and* $\nvdash_w [\Gamma' \mid \Delta']$.

As usual in canonical model techniques, we prove the crucial Truth Lemma:

Lemma 8.7 (Truth Lemma) *For every* $[\Gamma \mid \Delta] \in W^c$*:*

$$\psi \in \Gamma \quad \textit{iff} \quad \mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \psi.$$

We are now ready to prove the main result of this section:

Theorem 8.8 *The following holds:*

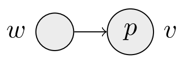
$$\begin{array}{llll} \textit{(1)} & \vdash_w [\Gamma \mid \Delta] & \textit{iff} & \Gamma \models_l \Delta \\ \textit{(2)} & \vdash_s [\Gamma \mid \Delta] & \textit{iff} & \Gamma \models_g \Delta. \end{array}$$

9 A Semantic Look Back

We use Theorem 8.8, stating that the logics **sBIL** and **wBIL** are respectively sound and complete with respect to the global and local consequence, to fill in the gaps of Sections 6 and 7 by proving the claims left pending there.

First, we can show the extensional difference of the two logics by proving Claim 6.1, which claims that **sBIL** $\not\subseteq$ **wBIL**:

Proof. [of Claim 6.1] On the one hand we obviously have that $\vdash_s [p \mid p]$ hence $\vdash_s [p \mid \neg{\sim}p]$ by DN_s. On the other hand we have that $p \not\models_l \neg{\sim}p$ as shown by the following model $\mathcal{M}_0$ where reflexive arrows are not depicted:



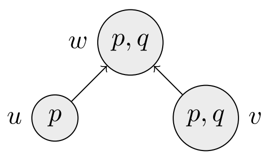
We clearly have $\mathcal{M}_0, v \Vdash p$. We also have $\mathcal{M}_0, v \nVdash \neg{\sim}p$ as $\mathcal{M}_0, v \Vdash {\sim}p$ because $\mathcal{M}_0, w \nVdash p$ and $w \leq v$. By Theorem 8.8 we obtain $\nvdash_w [p \mid \neg{\sim}p]$. $\square$

Second, we resolve Claim 7.5 that **sBIL** fails the deduction theorem.

Proof. [of Claim 7.5] We need to prove that $\nvdash_s [\emptyset \mid p \to \neg{\sim}p]$. Consider the model $\mathcal{M}_0$ above. We have that $\mathcal{M}_0, v \nVdash p \to \neg{\sim}p$, hence $\not\models_g p \to \neg{\sim}p$. By Theorem 8.8 we obtain $\nvdash_s [\emptyset \mid p \to \neg{\sim}p]$. Since applying DN_s to $p \vdash_s p$ gives $\vdash_s [p \mid \neg{\sim}p]$, the deduction theorem does not hold for **sBIL**. $\square$

Lastly, we prove Claim 7.10 that **sBIL** fails the dual deduction theorem:

Proof. [of Claim 7.10] We need to prove that $\nvdash_s [p \mid q \vee \neg{\sim}{\sim}\, q]$. Consider the following model $\mathcal{M}_1$:



First we have that $\mathcal{M}_1 \Vdash p$. We also have $\mathcal{M}_1, u \nVdash q$ by definition of the valuation. But we also have $\mathcal{M}_1, u \nVdash \neg{\sim}{\sim}\, q$. In fact $\mathcal{M}_1, w \Vdash {\sim}{\sim}\, q$ as $v \leq w$ and $\mathcal{M}_1, v \nVdash {\sim}\, q$: its only predecessor is itself, and it forces q. Consequently we have $p \not\models_g q \vee \neg{\sim}{\sim}\, q$, which by Theorem 8.8 gives $\nvdash_s [p \mid q \vee \neg{\sim}{\sim}\, q]$. $\square$

10 Why Rauszer's Proofs are Erroneous

The existence of **sBIL** and **wBIL** justifies our use of the plural bi-intuitionistic logics. We now trace the effect of this bifurcation on Rauszer's works.

As far as we know, neither the existence of **wBIL** and **sBIL**, nor the distinction between them has been highlighted in the literature. While it was certainly not noted in Rauszer's works, it has to be acknowledged that Hiroakira Ono may have suspected something [23, p.7]. Our bifurcation is not important if we only focus on one of the logics and use properties only belonging to it. However if one confuses them by using properties of these logics that are not shared by both of them, then troubles arrive. Unfortunately, such a confusion is made in some of Rauszer's works. As a consequence, various important theorems are asserted with erroneous proofs. The most important of them is the theorem of strong completeness with respect to the Kripke semantics [20]. More precisely there are two proofs for this theorem. The first one [20, Lemma 2.3], is flawed because it ignores restrictions on the use of a lemma proved by Gabbay [7], and this is extraneous to the confusion between the logics **sBIL** and **wBIL**. The second one [20, Theorem 3.5], is a standard completeness proof, involving the construction of a canonical model. However, in this proof, some intermediate lemmas are proved using features which are distinct for these logics. For example the fact that $\vdash_s [\varphi \mid \neg{\sim}\varphi]$ holds is used in the proof of Lemma 3.1 [20], where it is erroneously claimed that a prime filter A is such that if $a \in A$ then $\neg{\sim} a \in A$ and hence ${\sim} a \notin A$. In addition, in the proof of point (3) of Lemma 3.3 [20] the deduction theorem is used implicitly as it relies on a proof provided by Thomason [26] which uses it. Thus, while the proof of Lemma 3.1 [20] suggests the logic used is **sBIL**, the proof of Lemma 3.3 indicates that it must be **wBIL**. Thus the proof of completeness given there, which relies on these two lemmas, is a proof for none of the logics discussed here.

Another strong completeness proof [18] suffers from the same confusion because it relies on the aforementioned completeness proofs [20]. Interestingly, some elements of this paper [18] were corrected [21], but the corrections do not suffice to fix the issue. More precisely, one side of the deduction theorem is changed from $\Gamma \vdash \varphi \to \psi$ to $\Gamma \vdash \neg{\sim}\varphi \to \psi$ [21], but this version also fails for **sBIL** and, in any case, the proofs [20] are not modified to handle the change.

In a nutshell, as the proofs of strong completeness for bi-intuitionistic logic given in Rauszer's PhD thesis [23] are taken from the articles mentioned above, we are left with no actual trace in Rauszer's papers of a correct proof of strong completeness of bi-intuitionistic logic with respect to the Kripke semantics defined. To the best of our knowledge such a proof has only been provided by Sano and Stell [25], but for a different axiomatization. So, our proofs are the first to ensure that Rauszer's axiomatization is strongly complete for the appropriate Kripke semantics in a non-ambiguous way: **sBIL** (**wBIL**) is strongly complete for global (local) semantic consequence in Kripke semantics.

Clearly, providing such a proof is necessary to set the record straight for Rauszer's axiomatization. Furthermore, when compared with the initial proofs, our proofs are useful for avoiding false conclusions hinted at by the former. Most importantly, two proofs of strong completeness [20] involve the construction of a *rooted* canonical model where by "rooted" we understand the following

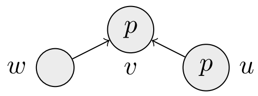
Definition 10.1 Let $\mathcal{F} = (W, \leq)$ be a BI-Kripke frame. We say that $\mathcal{F}$ is rooted if there is a $w \in W$ such that for every $v \in W$ we have $w \leq^* v$ (but since $\leq$ is reflexive and transitive we can replace $\leq^*$ with $\leq$).

The use of rooted models immediately implies that bi-intuitionistic logic is sound and complete with respect to the class of *rooted* BI-Kripke frames. However, we show that this result fails for both **sBIL** and **wBIL**! Specifically, $\sim p \vee \neg \sim p$ is valid on rooted frames but not valid on the full class of frames.

Lemma 10.2 *Let $\mathcal{F} = (W, \leq)$ be a rooted BI-Kripke frame. For any interpretation function I, we have that $(W, \leq, I) \Vdash \sim p \vee \neg \sim p$.*

Proof. Let r be the root of $\mathcal{F}$ and I an interpretation function, $\mathcal{M} = (W, \leq, I)$ and $w \in W$. As $\mathcal{F}$ is rooted we have that $r \leq w$. If $r \in I(p)$ then persistence and rootedness give $\mathcal{M}, v \Vdash p$ for every $v \in W$, hence $\mathcal{M}, w \Vdash \neg \sim p$. If $r \notin I(p)$ then we get $\mathcal{M}, w \Vdash \sim p$. In each case we obtain $\mathcal{M}, w \Vdash \sim p \vee \neg \sim p$. □

Thus the formula $\sim p \vee \neg \sim p$ is valid on the class of rooted BI-Kripke frames. Now we show that there is a BI-Kripke model $\mathcal{N}$ such that $\mathcal{N} \nVdash \sim p \vee \neg \sim p$. Consider the following model where reflexive arrows are omitted:



We have that $\mathcal{N}, u \nVdash \sim p$ as the only predecessor of u is itself and $\mathcal{N}, u \Vdash p$. Moreover we have that $\mathcal{N}, w \nVdash p$, hence $\mathcal{N}, v \Vdash \sim p$ which in turn implies $\mathcal{N}, u \nVdash \neg \sim p$. Consequently $\mathcal{N}, u \nVdash \sim p \vee \neg \sim p$.

It can be argued that Crolard [3, p.168] proved that **wBIL** is not complete for the class of rooted frames. But he does not make the distinction between the two logics presented here, nor pinpoint the flaws in Rauszer's proof.

Theorem 8.8 allows us to claim that $\nvdash_i [\emptyset \mid \sim p \vee \neg \sim p]$ for $i \in \{s, w\}$ as $\not\models_j \sim p \vee \neg \sim p$ for $j \in \{g, l\}$. From this, we conclude that neither **sBIL** nor **wBIL** is complete, with their corresponding semantic consequence, for the class of rooted frames: the formula $\sim p \vee \neg \sim p$ is a counterexample to such a claim.

11 Conclusion

Generalized Hilbert calculi effectively provide the tools to clarify the status of rules in axiomatic systems. The distinction between the two logics **sBIL** and **wBIL** can easily be tracked to the obvious difference between the rules DN_w and DN_s in the calculi defining them. Effectively, as in the modal case, different syntactic consequence relations stem from the traditional Hilbert calculus for bi-intuitionistic logic, formalized as generalized Hilbert calculi. The logics **wBIL** and **sBIL** are distinguishable on an extensional level in a similar way to **wK** and **sK**. The similarity with modal logic goes even further as the famous deduction theorem is not a property common to both **sBIL** and **wBIL**. As we have shown, the deduction theorem can be modified to hold in **sBIL**, and the dual deduction theorem holds in **wBIL**, but we have not yet found a modification of the dual deduction theorem for **sBIL**. So, on top of allowing one to clearly detect which logic satisfies the deduction theorem or its dual, generalized Hilbert calculi also prevent the confusions that existed in both the modal [11] and bi-intuitionistic case.

As we have shown, the logics **wBIL** and **sBIL**, respectively, have a local and global semantic counterpart on the class of BI-Kripke frames. Although quite common, this phenomenon finally clarifies the relation between the two logics. It also helps rectify the status of some properties of **sBIL** and **wBIL**, such as the fact that they are not strongly complete with respect to the class of rooted frames.

Finally, the difference between the two logics allows to look at the proof theory of bi-intuitionistic logic from a different angle. We conjecture that the various calculi which have been designed to capture bi-intuitionistic logic [8,9,10,14] are in fact sound and strongly complete for **wBIL**.

There are several directions for further work. First, the diversity of interpretations of the MP rule should be investigated. While we made a case of the multiplicity of interpretations (which we have not exhausted) of the rules DN and Nec, we did not question the shape of the rule MP. We could modify one of the generalized Hilbert calculi defined above to use a modified version of MP where the premisses would be $\emptyset \vdash A$ and $\emptyset \vdash A \to B$. This system would define a logic, but a weird one where $p, p \to q \vdash q$ would not be guaranteed to hold. A second direction, which we are exploring, leads to the algebraic treatment of **wBIL** and **sBIL** as consequence relations [6]. Third, the use of pairs $[\Gamma \mid \Delta]$ suggests a general treatment of logics that would capture both *derivability* and *refutability* calculi in one shot. Finding if such a general framework exists would require further investigations.

Related works: It has to be noted that Sano and Stell's axiomatization [25], when considered in a generalized Hilbert calculus context, also suffers from the same phenomenon as Rauszer's axiomatization. Their rule Mon$\prec$ can be interpreted in the same ways as DN: with a set of assumption in its premise, giving Mon$\prec_s$; or without, giving Mon$\prec_w$. The generalized Hilbert calculus involving the rule Mon$\prec_w$ (Mon$\prec_s$) corresponds to **wBIL** (**sBIL**).

Appendix

Proof. [of Lemma 5.2] **Monotonicity**: Assume $\Gamma \vdash_i \varphi$. Then there is a derivation π of $\Gamma \vdash \varphi$. We prove by induction on $l(\pi)$ that $\Gamma' \vdash_i \varphi$ with $\Gamma \subseteq \Gamma'$. If $l(\pi) = 1$ then either $\varphi \in \Gamma$ or $\varphi \in \mathcal{A}^I$. If $\varphi \in \Gamma$ then $\varphi \in \Gamma'$, hence $\Gamma' \vdash_i \varphi$. If

$\varphi \in \mathcal{A}^I$ then $\Gamma' \vdash_i \varphi$. If $l(\pi) > 1$ then we have to consider the last rule applied. if it is MP then we simply apply the induction hypothesis on the premises and apply the rule to obtain our result. In the case of DN_i we have to distinguish between the case where $i = s$ and $i = w$. If $i = s$ then we can simply use the induction hypothesis on the premises and then apply the rule. If $i = w$ then we simply use the given premise to obtain $\Gamma' \vdash_w \varphi$ as desired.

Compositionality: Assume $\Gamma \vdash_i \varphi$ and that $\Delta \vdash_i \gamma$ for every $\gamma \in \Gamma$. Then we have a derivation π of $\Gamma \vdash \varphi$. We show by induction on the length $l(\pi)$ of π that $\Delta \vdash_i \varphi$. If $l(\pi) = 1$ then either $\varphi \in \Gamma$, or $\varphi \in \mathcal{A}^I$. If $\varphi \in \Gamma$, we have $\Delta \vdash_i \varphi$ by assumption. If $\varphi \in \mathcal{A}^I$, then $\Delta \vdash_i \varphi$. If $l(\pi) > 1$ then consider the last rule applied. If it is MP then we can simply apply the induction hypothesis on the premises and then apply MP to obtain the required conclusion. If it is DN_i, then we must distinguish $i = s$ and $i = w$. If $i = s$, we apply the induction hypothesis on the premise and then the rule. If $i = w$, we apply appropriately the rule, i.e. from $\emptyset \vdash_w \varphi$ to $\Delta \vdash_w \varphi$, to obtain the desired result.

Structurality:Assume $\Gamma \vdash_i \varphi$. Then we have a derivation π of $\Gamma \vdash \varphi$. We will show by induction on $l(\pi)$ that $\Gamma^\sigma \vdash_i \varphi^\sigma$. If $l(\pi) = 1$ then either $\varphi \in \Gamma$ or $\varphi \in \mathcal{A}^I$. If $\varphi \in \Gamma$ then $\varphi^\sigma \in \Gamma^\sigma$, hence $\Gamma^\sigma \vdash \varphi^\sigma$. If $\varphi \in \mathcal{A}^I$ then $\varphi^\sigma \in \mathcal{A}^I$, hence $\Gamma^\sigma \vdash_i \varphi^\sigma$. If $l(\pi) > 1$ then consider the last rule applied. If it is MP then we apply the induction hypothesis on the premises and then apply MP to obtain the required conclusion. If it is DN_i then for both values of i we apply the induction hypothesis on the premise and then the rule. $\Box$

Proof. [of Lemma 5.3] Assume $\Gamma \vdash_i \varphi$, giving a derivation π with root $\Gamma \vdash \varphi$. We prove by induction on $l(\pi)$ that there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_i \varphi$. If $l(\pi) = 1$ then either $\varphi \in \Gamma$ or $\varphi \in \mathcal{A}^I$. If $\varphi \in \Gamma$, then $\{\varphi\} \subseteq \Gamma$ and $\varphi \vdash_i \varphi$. If $\varphi \in \mathcal{A}^I$, then $\emptyset \subseteq \Gamma$ and $\emptyset \vdash_i \varphi$. If $l(\pi) > 1$ then we consider the last rule applied. If the last rule is MP, then apply the induction hypothesis on the premises to obtain finite $\Gamma', \Gamma'' \subseteq \Gamma$ such that $\Gamma' \vdash_i \psi$ and $\Gamma'' \vdash_i \psi \to \varphi$. Theorem 5.2 delivers $\Gamma' \cup \Gamma'' \vdash_i \psi$ and $\Gamma' \cup \Gamma'' \vdash_i \psi \to \varphi$. Thus MP can be applied to get $\Gamma' \cup \Gamma'' \vdash_i \varphi$, where $\Gamma' \cup \Gamma'' \subseteq \Gamma$ is finite. If the last rule is DN_i, then $i = s$ or $i = w$. If $i = s$, we apply the induction hypothesis on the premise and then the rule. If $i = w$, we apply appropriately the rule to obtain the desired result. $\Box$

Proof. [of Proposition 7.2]

($\Rightarrow$) Assume $\vdash_i [\emptyset \mid (\varphi \prec \psi) \to \chi]$, i.e. $\emptyset \vdash_i (\varphi \prec \psi) \to \chi$. From it we can easily obtain $\emptyset \vdash_i ((\varphi \prec \psi) \vee \psi) \to (\chi \vee \psi)$. But as we have $\emptyset \vdash_i \varphi \to ((\varphi \prec \psi) \vee \psi)$ we get $\emptyset \vdash_i \varphi \to (\chi \vee \psi)$, hence $\vdash_i [\emptyset \mid \varphi \to (\chi \vee \psi)]$.

($\Leftarrow$) Assume $\vdash_i [\emptyset \mid \varphi \to (\psi \vee \chi)]$, i.e. $\emptyset \vdash_i \varphi \to (\psi \vee \chi)$. First we have, as an instance of the axiom RA_{11}, $\emptyset \vdash_i (\varphi \prec \psi) \to (\chi \vee ((\varphi \prec \psi) \prec \chi))$. But we have that $\emptyset \vdash_i ((\varphi \prec \psi) \prec \chi) \leftrightarrow (\varphi \prec (\psi \vee \chi))$, so we obtain that $\emptyset \vdash_i (\chi \vee ((\varphi \prec \psi) \prec \chi)) \to (\chi \vee (\varphi \prec (\psi \vee \chi)))$. Thus $\emptyset \vdash_i (\varphi \prec \psi) \to (\chi \vee (\varphi \prec (\psi \vee \chi)))$. However we have that $\emptyset \vdash_i (\varphi \prec (\psi \vee \chi)) \to \sim (\varphi \to (\psi \vee \chi))$. And as we have $\emptyset \vdash_i \varphi \to (\psi \vee \chi)$ by

DN_i we obtain $\emptyset \vdash_i \neg\sim(\varphi \to (\psi \vee \chi))$. Consequently we can obtain that $\emptyset \vdash_i (\varphi \prec (\psi \vee \chi)) \to \bot$, hence $\emptyset \vdash_i (\chi \vee (\varphi \prec (\psi \vee \chi))) \to \chi$. This finally implies $\emptyset \vdash_i (\varphi \prec \psi) \to \chi$, hence $\vdash_i [\emptyset \mid (\varphi \prec \psi) \to \chi]$.

□

Proof. [of Theorem 7.3]

($\Leftarrow$) Assume $\vdash_w [\Gamma \mid \varphi \to \psi]$, i.e. $\Gamma \vdash_w \varphi \to \psi$. Then by monotonicity we obtain $\Gamma, \varphi \vdash_w \varphi \to \psi$. Moreover we have that $\Gamma, \varphi \vdash_w \varphi$ as $\varphi \in \Gamma \cup \{\varphi\}$. So by MP we obtain $\Gamma, \varphi \vdash_w \psi$, hence $\vdash_w [\Gamma, \varphi \mid \psi]$.

($\Rightarrow$) Assume $\vdash_w [\Gamma, \varphi \mid \psi]$, i.e. $\Gamma, \varphi \vdash_w \psi$ giving a derivation π of $\Gamma, \varphi \vdash \psi$. We show by induction on the length of π that $\Gamma \vdash_w \varphi \to \psi$. If $l(\pi) = 1$ then either $\psi \in \Gamma \cup \{\varphi\}$ or $\psi \in \mathcal{A}^I$. If $\varphi = \psi$ then we clearly have $\Gamma \vdash_w \varphi \to \psi$. If $\psi \in \Gamma$ then we can deduce $\Gamma \vdash_w \varphi \to \psi$ from the fact that we have $\emptyset \vdash_w p \to (q \to p)$. If $\psi \in \mathcal{A}^I$ then with a similar reasoning $\Gamma \vdash_w \varphi \to \psi$. If $l(\pi) > 1$ then consider the last rule applied. The case of the rule MP is treated as follows. Use the induction hypothesis on the premises of the rule and note that $\emptyset \vdash_w (p \to q) \to ((p \to (q \to r)) \to (p \to r))$: using MP several times one arrives at the establishment of $\Gamma \vdash_w \varphi \to \psi$. If the last rule is DN_w, we have a derivation of $\emptyset \vdash \chi$, so we can apply DN_w to obtain $\emptyset \vdash_w \neg\sim\chi$. Then we can use the fact that $\emptyset \vdash_w p \to (q \to p)$ to obtain $\emptyset \vdash_w \varphi \to \neg\sim\chi$. By monotonicity we obtain $\Gamma \vdash_w \varphi \to \neg\sim\chi$.

□

For the proof of Theorem 7.7 we need the following claim:

Claim .1 $\emptyset \vdash_s \neg(\lambda_1 \prec \lambda_2) \to (\sim\lambda_2 \to \sim\lambda_1)$

Proof. We have $\emptyset \vdash_s \lambda_1 \to (\lambda_2 \vee (\lambda_1 \prec \lambda_2))$. The rule below is derivable in both systems:

$$\frac{\emptyset \vdash \varphi \to \psi}{\emptyset \vdash (\varphi \prec \chi) \to (\psi \prec \chi)} \prec_{Mon}$$

Indeed, given that $\emptyset \vdash_i \psi \to (\chi \vee (\psi \prec \chi))$ and $\emptyset \vdash_i \varphi \to \psi$, we get $\emptyset \vdash_i \varphi \to (\chi \vee (\psi \prec \chi))$, hence $\emptyset \vdash_i (\varphi \prec \chi) \to (\psi \prec \chi)$ by Proposition 7.2. We can apply $\prec Mon$ to obtain $\emptyset \vdash_s (\top \prec (\lambda_2 \vee (\lambda_1 \prec \lambda_2))) \to \sim\lambda_1$. Next we prove that $\emptyset \vdash_s (\sim\lambda_2 \wedge \neg(\lambda_1 \prec \lambda_2)) \to \sim(\lambda_2 \vee (\lambda_1 \prec \lambda_2))$ to obtain that $\emptyset \vdash_s (\sim\lambda_2 \wedge \neg(\lambda_1 \prec \lambda_2)) \to \sim\lambda_1$, and hence $\emptyset \vdash_s \neg(\lambda_1 \prec \lambda_2) \to (\sim\lambda_2 \to \sim\lambda_1)$. First, $\emptyset \vdash_s \top \to ((\lambda_2 \vee (\lambda_1 \prec \lambda_2)) \vee (\top \prec (\lambda_2 \vee (\lambda_1 \prec \lambda_2))))$ is an instance of an axiom. Then by associativity of disjunction we obtain $\emptyset \vdash_s \top \to (\lambda_2 \vee ((\lambda_1 \prec \lambda_2) \vee (\top \prec (\lambda_2 \vee (\lambda_1 \prec \lambda_2)))))$. By Proposition 7.2 we get $\emptyset \vdash_s \sim\lambda_2 \to ((\lambda_1 \prec \lambda_2) \vee (\top \prec (\lambda_2 \vee (\lambda_1 \prec \lambda_2))))$. Consequently we easily obtain $\emptyset \vdash_s (\sim\lambda_2 \wedge \neg(\lambda_1 \prec \lambda_2)) \to (\top \prec (\lambda_2 \vee (\lambda_1 \prec \lambda_2)))$, i.e.
$\emptyset \vdash_s (\sim\lambda_2 \wedge \neg(\lambda_1 \prec \lambda_2)) \to \sim(\lambda_2 \vee (\lambda_1 \prec \lambda_2))$. □

Proof. [of Theorem 7.7]

($\Rightarrow$) Assume that $\vdash_s [\Gamma, \varphi \mid \psi]$, i.e. that we have a derivation π of $\Gamma, \varphi \vdash \psi$. We reason by induction on the length of π. If $l(\pi) = 1$ then two cases are possible. If the rule applied is Ax then we get $\emptyset \vdash_s \psi$, and as we have that

$\emptyset \vdash_s \psi \to ((\neg\sim)^n\varphi \to \psi)$ for any $n \in \mathbb{N}$ we obtain by MP: $\emptyset \vdash_s (\neg\sim)^n\varphi \to \psi$. By Theorem 5.2 we get $\Gamma \vdash_s (\neg\sim)^n\varphi \to \psi$. If the rule applied is El then either $\psi = \varphi$ and then we get $\emptyset \vdash_s \varphi \to \varphi$ and hence $\Gamma \vdash_s \varphi \to \varphi$, where in the antecedent of the implication $\varphi = (\neg\sim)^0\varphi$. If $l(\pi) \geq 1$ then two cases have to be considered. If the last rule applied is MP then we have by induction hypothesis $\Gamma \vdash_s (\neg\sim)^l\varphi \to \chi$ and $\Gamma \vdash_s (\neg\sim)^m\varphi \to (\chi \to \psi)$ for some χ, $m, l \in \mathbb{N}$. As we have that $\emptyset \vdash_s (\lambda_1 \to \lambda_2) \to ((\lambda_1 \to (\lambda_2 \to \lambda_3)) \to (\lambda_1 \to \lambda_3))$ and $\emptyset \vdash_s \neg\sim\lambda \to \lambda$ we obtain that $\Gamma \vdash_s \neg\sim^n\varphi \to \chi$ for $n = max(m, l)$.

If the last rule applied is DN_s then we get by induction hypothesis that $\Gamma \vdash_s (\neg\sim)^n\varphi \to \chi$. If we prove that $\emptyset \vdash_s \neg\sim(\lambda_1 \to \lambda_2) \to (\neg\sim\lambda_1 \to \neg\sim\lambda_2)$ holds then we can reach our goal by applying DN_s on $\Gamma \vdash_s (\neg\sim)^n\varphi \to \chi$ to obtain $\Gamma \vdash_s \neg\sim((\neg\sim)^n\varphi \to \chi)$ and finally $\Gamma \vdash_s (\neg\sim)^{n+1}\varphi \to \neg\sim\chi$ by MP, hence $\vdash_s [\Gamma \mid (\neg\sim)^{n+1}\varphi \to \neg\sim\chi]$. Let us thus prove $\emptyset \vdash_s \neg\sim(\lambda_1 \to \lambda_2) \to (\neg\sim\lambda_1 \to \neg\sim\lambda_2)$. First note that $\emptyset \vdash_s (\lambda_1 \prec \lambda_2) \to\sim(\lambda_1 \to \lambda_2)$ as it is an instance of the axiom A_{12}. By using A_{10} and MP we obtain $\emptyset \vdash_s \neg\sim(\lambda_1 \to \lambda_2) \to \neg(\lambda_1 \prec \lambda_2)$. Thus, using Claim .1, which can be found just above, we can obtain $\emptyset \vdash_s \neg\sim(\lambda_1 \to \lambda_2) \to (\sim\lambda_2 \to\sim\lambda_1)$. We can instantiate A_{10} again to obtain $\emptyset \vdash_s (\sim\lambda_2 \to\sim\lambda_1) \to (\neg\sim\lambda_1 \to \neg\sim\lambda_2)$, and use this fact with the previous result to finally get $\emptyset \vdash_s \neg\sim(\lambda_1 \to \lambda_2) \to (\neg\sim\lambda_1 \to \neg\sim\lambda_2)$.

($\Leftarrow$) Straightforward use of the rules DN_s and MP with Theorem 5.2.

□

Proof. [of Lemma 8.3] We reason by induction on φ and only show the cases for the added operators:

- $\varphi :=\sim\psi$: $\mathcal{M}, w \Vdash\sim\psi$ then there is a $u \leq w$ such that $\mathcal{M}, u \nVdash \psi$. By transitivity we have $u \leq v$, so there is a $u \leq v$ such that $\mathcal{M}, u \nVdash \psi$. Thus $\mathcal{M}, v \Vdash\sim\psi$.
- $\varphi := \chi \prec \psi$: $\mathcal{M}, w \Vdash \chi \prec \psi$ then there is a $u \leq w$ such that $\mathcal{M}, u \Vdash \chi$ and $\mathcal{M}, u \nVdash \psi$. By transitivity we have $u \leq v$, so there is a $u \leq v$ such that $\mathcal{M}, u \Vdash \chi$ and $\mathcal{M}, u \nVdash \psi$. Thus $\mathcal{M}, v \Vdash \chi \prec \psi$.

□

Proof. [of Lemma 8.6] We start by extending the set Γ to a prime theory Γ' in $\mathcal{L}$ by successive steps. More precisely we create a chain of extensions $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 ...$, where $\Gamma_0 = \Gamma$ and $\Gamma' = \bigcup_{k \geq 0} \Gamma_k$. In fact, we take an enumeration of all formulae of $\mathcal{F}orm_{BI}$ and we define Γ_n by induction on $n \in \mathbb{N}$ in the following way:

- $n = 0 : \Gamma_0 = \Gamma$;
- $n \geq 0$: let $\psi_1 \vee \psi_2$ be the first disjunctive sentence of $\mathcal{L}$ that has not yet been treated such that $\vdash_w [\Gamma_n \mid \psi_1 \vee \psi_2]$. Define:

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\psi_1\}, & \text{if } \nvdash_w [\Gamma_n, \psi_1 \mid \Delta] \\ \Gamma_n \cup \{\psi_2\}, & \text{otherwise} \end{cases}$$

We first show that $\nvdash_w [\Gamma' \mid \Delta]$. If we show that $\nvdash_w [\Gamma_n \mid \Delta]$ for every $n \in \mathbb{N}$ then we are done. Let us show this statement by induction on n. The base case holds by assumption as $\Gamma_0 = \Gamma$. For the inductive step we have to show that $\nvdash_w [\Gamma_n, \psi_i \mid \Delta]$. But this is obvious as it cannot both be the case that $\vdash_w [\Gamma_n, \psi_1 \mid \Delta]$ and $\vdash_w [\Gamma_n, \psi_2 \mid \Delta]$, otherwise we would have $\vdash_w [\Gamma_n \mid \Delta]$ as $\vdash_w [\Gamma_n \mid \psi_1 \vee \psi_2]$.

Second, we need to show some properties of Γ':

(i) *Consistency*: Γ' is consistent as $\nvdash_w [\Gamma' \mid \Delta]$.

(ii) *Primeness*: Let $\psi_1 \vee \psi_2 \in \Gamma'$ and k the least number such that $\vdash_w [\Gamma_k \mid \psi_1 \vee \psi_2]$. At stage k this $\psi_1 \vee \psi_2$ has not been treated and is treated eventually at a stage $j \geq k$. Then we get that $\psi_1 \in \Gamma_{j+1}$ or $\psi_2 \in \Gamma_{j+1}$, hence $\psi_1 \in \Gamma'$ or $\psi_2 \in \Gamma'$.

(iii) *Closure under deducibility*: Let ψ be a formula such that $\vdash_w [\Gamma' \mid \psi]$. Then $\vdash_w [\Gamma' \mid \psi \vee \psi]$ and as Γ' is prime we get that $\psi \in \Gamma'$.

Third we define $\Delta' = \{\psi \mid \nvdash_w [\Gamma' \mid \psi]\}$. First note that $\Delta \subseteq \Delta'$. Second we obtain that $\mathcal{F}orm_{BI} \setminus \Gamma' = \Delta'$ by definition of derivation and the closure under deducibility of Γ'. So $[\Gamma' \mid \Delta']$ is complete. We obviously obtain that this pair is unprovable: assume otherwise, then there is a finite $\Delta_0 \subseteq \Delta'$ such that $\vdash_w [\Gamma' \mid \bigvee \Delta_0]$, but as Γ' is closed under deducibility and prime we obtain that there is $\psi \in \Delta'$ such that $\psi \in \Gamma'$, which is a contradiction.

So $[\Gamma' \mid \Delta']$ is a complete pair with $\nvdash_w [\Gamma' \mid \Delta']$ and $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. $\square$

Proof. [of Lemma 8.7] By induction on ψ. We only consider the case for $\prec$:

- $\psi := \psi_1 \prec \psi_2$: ($\Rightarrow$) Assume $\psi_1 \prec \psi_2 \in \Gamma$. We claim that $\nvdash_w [\psi_1 \mid \psi_2, \Delta]$. Suppose it is not the case. Then by definition there is a finite $\Delta_f \subseteq \Delta$ such that $\psi_1 \vdash_w \psi_2 \vee \bigvee \Delta_f$, hence $\vdash_w [\psi_1 \mid \psi_2 \vee \bigvee \Delta_f]$. By Theorem 7.3 we thus obtain $\vdash_w [\emptyset \mid \psi_1 \to (\psi_2 \vee \bigvee \Delta_f)]$. And then by Proposition 7.2 we obtain that $\vdash_w [\emptyset \mid (\psi_1 \prec \psi_2) \to \bigvee \Delta_f]$. But as $\psi_1 \prec \psi_2 \in \Gamma$ and Γ is closed under deducibility we get that $\bigvee \Delta_f \in \Gamma$, which leads to an obvious contradiction. So $\nvdash_w [\psi_1 \mid \psi_2, \Delta]$. Thus by Lemma 8.6 there are $\Gamma' \supseteq \{\psi_1\}$ and $\Delta' \supseteq \Delta \cup \{\psi_2\}$ such that $[\Gamma' \mid \Delta']$ is complete and $\nvdash_w [\Gamma' \mid \Delta']$. Note that $\psi_1 \in \Gamma'$ and $\psi_2 \notin \Gamma'$, hence $\mathcal{M}^c, [\Gamma' \mid \Delta'] \Vdash \psi_1$ and $\mathcal{M}^c, [\Gamma' \mid \Delta'] \nVdash \psi_2$ by induction hypothesis. But we have that $\Delta \subseteq \Delta'$, which implies by completeness that $\Gamma' \subseteq \Gamma$. So $[\Gamma' \mid \Delta'] \leq^c [\Gamma \mid \Delta]$. Consequently $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \psi_1 \prec \psi_2$. ($\Leftarrow$) Assume $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \psi_1 \prec \psi_2$. Assume for reductio that $\psi_1 \prec \psi_2 \notin \Gamma$. Then $\nvdash_w [\Gamma \mid \psi_1 \prec \psi_2]$. Note that every $\Gamma' \subseteq \Gamma$ is such that $\psi_1 \prec \psi_2 \notin \Gamma'$. And as $\vdash_w [\emptyset \mid \psi_1 \to (\psi_2 \vee (\psi_1 \prec \psi_2))]$ we get for every $\Gamma' \subseteq \Gamma$ such that $[\Gamma' \mid \Delta'] \in W^c$ for some Δ', if $\psi_1 \in \Gamma'$ then $\psi_2 \in \Gamma'$ as Γ' is prime. By induction hypothesis we get that for every such Γ', if $\mathcal{M}^c, [\Gamma' \mid \Delta'] \Vdash \psi_1$ then $\mathcal{M}^c, [\Gamma' \mid \Delta'] \Vdash \psi_2$. This contradicts our assumption $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \psi_1 \prec \psi_2$. $\square$

Proof. [of Theorem 8.8] Soundness is straightforward so let us prove (1):

($\Leftarrow$) Here we prove *completeness*. Assume $\nvdash_w [\Gamma \mid \Delta]$. Lemma 8.6 gives us a complete $[\Gamma' \mid \Delta']$ such that $\nVdash_w [\Gamma' \mid \Delta']$, where $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. Moreover there is no $\delta \in \Delta$ such that $\delta \in \Gamma'$, so by Lemma 8.7 we obtain that in the canonical model of Definition 8.5 the following holds: $\mathcal{M}^c, [\Gamma' \mid \Delta'] \nVdash \delta$ for every $\delta \in \Delta$, while $\mathcal{M}^c, [\Gamma' \mid \Delta'] \Vdash \Gamma$. Consequently, we have that $\Gamma \not\models_l \Delta$.

Then we can prove (2):

($\Leftarrow$) Here we prove *completeness*. Assume $\nvdash_s [\Gamma \mid \Delta]$. We show that $\Gamma \not\models_g \Delta$. Note that $\nvdash_s [(\neg{\sim})^\omega\Gamma \mid \Delta]$ from Theorem 7.7. Thus, we get $\nvdash_w [(\neg{\sim})^\omega\Gamma \mid \Delta]$ by Theorem 6.2. By the argument used in the strong completeness of **wBIL** we know that there is a pair $[((\neg{\sim})^\omega\Gamma)^* \mid \Delta']$ in the canonical model of Definition 8.5 such that $(\neg{\sim})^\omega\Gamma \subseteq ((\neg{\sim})^\omega\Gamma)^*$ and $\Delta \subseteq \Delta'$. Lemma 8.7 tells us that for all $\delta \in \Delta$ we have $\mathcal{M}^c, [((\neg{\sim})^\omega\Gamma)^* \mid \Delta'] \nVdash \delta$ and $\mathcal{M}^c, [((\neg{\sim})^\omega\Gamma)^* \mid \Delta'] \Vdash \neg{\sim}^\omega\Gamma$.

To obtain a proof of $\Gamma \not\models_g \Delta$ we need a Γ-model that has one point that is not a δ-point for all $\delta \in \Delta$. To do so we restrict the canonical model, on the point described above, to obtain a Γ-model. We define $\mathcal{M}^c_\Gamma = (W^c_\Gamma, \leq^c_\Gamma, I^c_\Gamma)$, where $W^c_\Gamma = \{[\Delta_1 \mid \Delta_2] \in W^c \mid$ there is a chain $[((\neg{\sim})^\omega\Gamma)^* \mid \Delta']R_1...R_n[\Delta_1 \mid \Delta_2]$, where $R_j \in \{\leq, \geq\}$ for $j \in \mathbb{N}\}$, and I^c_Γ and $\leq^c_\Gamma$ are restrictions of respectively I^c and $\leq^c$ to W^c_Γ. The notion of bisimulation developed by de Groot and Pattinson [5], gives us that $(\mathcal{M}^c, [((\neg{\sim})^\omega\Gamma)^* \mid \Delta'])$ and $(\mathcal{M}^c_\Gamma, [((\neg{\sim})^\omega\Gamma)^* \mid \Delta'])$ are bisimilar, hence modally equivalent. Thus we have that $\mathcal{M}^c_\Gamma, [((\neg{\sim})^\omega\Gamma)^* \mid \Delta] \nVdash \delta$ for every $\delta \in \Delta$, and $\mathcal{M}^c_\Gamma, [((\neg{\sim})^\omega\Gamma)^* \mid \Delta] \Vdash (\neg{\sim})^\omega\Gamma$. It remains to prove that $\mathcal{M}^c_\Gamma$ is a Γ-model. Let $[\Delta_1 \mid \Delta_2] \in \mathcal{M}^c_\Gamma$. By definition there is a chain $[((\neg{\sim})^\omega\Gamma)^* \mid \Delta']R_1...R_n[\Delta_1 \mid \Delta_2]$ such that $R_j \in \{\leq, \geq\}$ for every $j \in \{1, ..., n\}$. We now need the following Claim .2 to conclude that $\mathcal{M}^c_\Gamma, [\Delta_1 \mid \Delta_2] \Vdash (\neg{\sim})^\omega\Gamma$. In particular we obtain $\mathcal{M}^c_\Gamma, [\Delta_1 \mid \Delta_2] \Vdash \Gamma$.

Claim .2 *For every chain* $[((\neg{\sim})^\omega\Gamma)^* \mid \Delta']R_1...R_n[\Psi_1 \mid \Psi_2]$ *we have that* $\mathcal{M}^c_\Gamma, [\Psi_1 \mid \Psi_2] \Vdash (\neg{\sim})^\omega\Gamma$.

Proof. Let $C = [((\neg{\sim})^\omega\Gamma)^* \mid \Delta']R_1...R_n[\Psi_1 \mid \Psi_2]$ be a chain. We prove that $\mathcal{M}^c_\Gamma, [\Psi_1 \mid \Psi_2] \Vdash (\neg{\sim})^\omega\Gamma$ by induction on the length l of C:

- $l = 0$: then $[\Psi_1 \mid \Psi_2] = [((\neg{\sim})^\omega\Gamma)^* \mid \Delta']$ and consequently $\mathcal{M}^c_\Gamma, [((\neg{\sim})^\omega\Gamma)^* \mid \Delta'] \Vdash (\neg{\sim})^\omega\Gamma$ by Lemma 8.7;
- $l = n + 1$: if R_{n+1} is $\leq$ then there is $[\Pi_1 \mid \Pi_2]$ such that $[\Pi_1 \mid \Pi_2] \leq [\Psi_1 \mid \Psi_2]$. By induction hypothesis we get $\mathcal{M}^c_\Gamma, [\Pi_1 \mid \Pi_2] \Vdash (\neg{\sim})^\omega\Gamma$ and consequently, by Lemma 8.3 $\mathcal{M}^c_\Gamma, [\Psi_1 \mid \Psi_2] \Vdash (\neg{\sim})^\omega\Gamma$. If R_{n+1} is $\geq$ then there is $[\Pi_1 \mid \Pi_2]$ such that $[\Pi_1 \mid \Pi_2] \geq [\Psi_1 \mid \Psi_2]$. By induction hypothesis we get $\mathcal{M}^c_\Gamma, [\Pi_1 \mid \Pi_2] \Vdash (\neg{\sim})^\omega\Gamma$. Note that $\neg{\sim}(\neg{\sim})^\omega\Gamma = (\neg{\sim})^\omega\Gamma$, so $\mathcal{M}^c_\Gamma, [\Pi_1 \mid \Pi_2] \Vdash \neg{\sim}(\neg{\sim})^\omega\Gamma$. We easily obtain $\mathcal{M}^c_\Gamma, [\Psi_1 \mid \Psi_2] \Vdash (\neg{\sim})^\omega\Gamma$.

□

□

References

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