*-Continuity vs. Induction: Divide and Conquer

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Abstract

The Kleene star can be axiomatised in two ways: inductively, as a fixpoint, or as the ω-iteration of multiplications. The latter is called *-continuity and is stronger than the former: not every Kleene algebra is *-continuous. In the language of only multiplication, union, and Kleene star, however, the (in)equational atomic theory (logic) of *-continuous Kleene algebras coincides with the one of all Kleene algebras (Kozen, 1994). The situation changes dramatically when one adds division operations. As shown by Buszkowski (2007), then the logic with *-continuity becomes Π_1^0-hard and therefore strictly stronger than the inductive one. This result, however, is not constructive, i.e., does not yield a formula distingushing them.
Our contribution is threefold. First, we present an example of Kleene algebra with divisions and intersection, which is not *-continuous. Second, we present a formula which makes Buszkowski's result constructive (see above). Third, we show that the calculus for *-continuity is incomplete w.r.t. more specific relational and language models, in the fragment with divisions, multiplication, intersection, and Kleene star. The choice of this fragment is natural, since union or the unit constant are known to yield incompleteness even without Kleene star.

Keywords: Kleene star, infinitary action logic, action logic, residuated Kleene lattice, *-continuity, algebra of formal languages, relational algebra

1 Introduction

1.1 Residuated Kleene Lattices

We start with the definition of residuated Kleene lattice (RKL), or action lattice, following Kozen [17] and Buszkowski [5]. The notion of RKL is a combination of action algebras, or action semilattices, by Pratt [32], and residuated lattices studied by Ono [27] and others as models for substructural logics (*i.e.,* logics lacking structural rules of contraction, weakening, and permutation). In comparison with RKL's, action algebras lack meet ($\wedge$) and residuated lattices

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lack the most intriguing operation, the Kleene star. In RKL's, we have the full set of operations.

As a matter of notation, we follow the tradition of Lambek [21] and denote residuals as divisions ($\backslash$, $/$), rather than directed implications ($\to$, $\leftarrow$). For simplicity, we also do not use the zero constant in our algebraic definitions and logical calculi (examples of RKL's we present here, however, have a zero).

Definition 1.1 A structure $(\mathfrak{A}; \preceq, \vee, \wedge, \cdot, \mathbf{1}, \backslash, /, {}^*)$ is a *residuated Kleene lattice* (RKL), if the following holds.

(i) $(\mathfrak{A}; \preceq, \vee, \wedge)$ is a lattice (*i.e.,* $\preceq$ is a preorder, $x \vee y = \sup\{x, y\}$, $x \wedge y = \inf\{x, y\}$, where sup and inf are taken w.r.t. $\preceq$ and should exist).

(ii) $(\mathfrak{A}; \cdot, \mathbf{1})$ is a monoid (*i.e.,* $\cdot$ is an associative operation and $\mathbf{1}$ is its unit).

(iii) $\backslash$ and $/$ are residuals for $\cdot$ w.r.t. $\preceq$, *i.e.,*

$$y \preceq x \backslash z \iff x \cdot y \preceq z \iff x \preceq z / y$$

In other words, $x \backslash z = \max\{y \mid x \cdot y \preceq z\}$ and $z / y = \max\{x \mid x \cdot y \preceq z\}$, where max is taken w.r.t. $\preceq$. The structure $\mathfrak{A}$ is residuated if all these maxima exist. (Partially ordered residuated semigroups are algebraic models for the Lambek calculus [21].)

(iv) a^* is the smallest fixpoint of $x \mapsto \mathbf{1} \vee a \cdot x$, *i.e.,* $\mathbf{1} \preceq a^*$, $a \cdot a^* \preceq a^*$, and if $\mathbf{1} \preceq b$ and $a \cdot b \preceq b$, then $a^* \preceq b$.

As shown by Pratt [32], in the residuated situation there is no difference between left and right RKL's (cf. Kozen [15] for an example where a^* is a fixpoint for $x \mapsto \mathbf{1} \vee a \cdot x$, but not for $x \mapsto \mathbf{1} \vee x \cdot a$).

Moreover, Pratt [32] shows that in the presence of division operations the fixpoint condition (iv) can be reformulated. Namely, the 'smallest' half of (iv) is replaced by one axiom which Pratt called *pure induction:* $(a \backslash a)^* = a \backslash a$, and the monotonicity principle: if $a \preceq b$, then $a^* \preceq b^*$.[2] Pure induction is particularly easy to check. For the other half of (iv) Pratt suggests a symmetric version: $\mathbf{1} \preceq a^*$, $a \preceq a^*$, $a^* \cdot a^* \preceq a^*$.

Further we use the notation x^n, defined inductively: $x^0 = \mathbf{1}$, $x^{n+1} = x^n \cdot x$.

Definition 1.2 An RKL $\mathfrak{A}$ is **-continuous,* if $x^* = \sup\{x^n \mid n \in \omega\}$, where supremum is taken w.r.t. $\preceq$ and ω denotes the set of all natural numbers. (In particular, this implies that all such suprema exist.)

Notice that here the usual definition of *-continuity, $y \cdot x^* \cdot z = \sup\{y \cdot x^n \cdot z \mid n \in \omega\}$, here can be simpified, since in the presence of division operations multiplication distributes over infinite joins (suprema). In presence of *-continuity, axiom (iv) becomes redundant.

[2] Since in lattices $a \preceq b$ iff $b = a \vee b$, monotonicity can be reformulated as an (in)equation: $a^* \preceq (a \vee b)^*$. Pratt also does provides such a reformulation of Lambek-style axioms for division operations. This yields a purely (in)equational axiomatisation of action algebras, *i.e.,* the fact that the class of action algebras is a finitely based variety [32, Theorem 7].

1.2 Action Logic and Infinitary Action Logic

In this section we define logical calculi which correspond to algebraic structures defined before. We start with **MALC**, the multiplicative-additive Lambek calculus, [3] which is obtained from the Lambek calculus with the unit [22] by extending it with additive conjunction and disjunction from Girard's linear logic [9], corresponding to meet and join.

Formulae of **MALC** are built from a countable set of variables $\{p, q, r, \ldots\}$ and the unit constant $\mathbf{1}$ using five binary connectives: $\cdot$, $\backslash$, $/$, $\wedge$, $\vee$. Sequents of **MALC** are expressions of the form $\Pi \to A$, where A is a formula and Π is a finite (possibly empty) linearly ordered sequence of formulae.

Axioms and rules of **MALC** are as follows:

$$\frac{}{A \to A}\ (\mathrm{id}) \qquad \frac{\Gamma, \Delta \to C}{\Gamma, \mathbf{1}, \Delta \to C}\ (\mathbf{1} \to) \qquad \frac{}{\to \mathbf{1}}\ (\to \mathbf{1})$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \cdot B, \Delta \to C}\ (\cdot \to) \qquad \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \cdot B}\ (\to \cdot)$$

$$\frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, \Pi, A \setminus B, \Delta \to C}\ (\setminus \to) \qquad \frac{A, \Pi \to B}{\Pi \to A \setminus B}\ (\to \setminus)$$

$$\frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, B \,/\, A, \Pi, \Delta \to C}\ (/ \to) \qquad \frac{\Pi, A \to B}{\Pi \to B \,/\, A}\ (\to /)$$

$$\frac{\Gamma, A_1, \Delta \to C \quad \Gamma, A_2, \Delta \to C}{\Gamma, A_1 \vee A_2, \Delta \to C}\ (\vee \to) \qquad \frac{\Pi \to A_i}{\Pi \to A_1 \vee A_2}\ (\to \vee)_i,\ i = 1, 2$$

$$\frac{\Gamma, A_i, \Delta \to C}{\Gamma, A_1 \wedge A_2, \Delta \to C}\ (\wedge \to)_i,\ i = 1, 2 \qquad \frac{\Pi \to A_1 \quad \Pi \to A_2}{\Pi \to A_1 \wedge A_2}\ (\to \wedge)$$

$$\frac{\Pi \to A \quad \Gamma, A, \Delta \to C}{\Gamma, \Pi, \Delta \to C}\ (\mathrm{cut})$$

(The cut rule is eliminable by straightforward induction, as in the original Lambek calculus [21].)

This system is extended with the Kleene star * (as a unary connective) in two ways, depending on whether we want *-continuity.

The *-continuous case corresponds to *infinitary action logic* [6], $\mathbf{ACT}_\omega$, which is obtained from **MALC** by adding the following rules:

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[3] Sometimes **MALC** is called "full Lambek calculus," which sounds a bit offensive for the original, multiplicative-only system.

$$\frac{(\Gamma, A^n, \Delta \to C)_{n \geq 0}}{\Gamma, A^*, \Delta \to C}\ ({}^* \to)_\omega \qquad \frac{\Pi_1 \to A \quad \ldots \quad \Pi_n \to A}{\Pi_1, \ldots, \Pi_n \to A^*}\ (\to {}^*)_n,\ n \geq 0$$

Derivations in $\mathbf{ACT}_\omega$ are infinite, but well-founded trees with infinite branching at instances of $({}^* \to)_\omega$. Cut elimination for $\mathbf{ACT}_\omega$ was shown by Palka [29] using transfinite induction.

The system corresponding to the inductive definition of *, *action logic* $\mathbf{ACT}$, is obtained from $\mathbf{MALC}$ (with (cut)) by adding the following rules [5]:

$$\frac{}{\to A^*} \qquad \frac{}{A, A^* \to A^*} \qquad \frac{}{A^*, A \to A^*}$$

$$\frac{A, B \to B}{A^*, B \to B} \qquad \frac{B, A \to B}{B, A^* \to B}$$

These rules are not good Gentzen-style sequent rules, and cut in this formulation of $\mathbf{ACT}$ is not eliminable. In fact, no cut-free sequent calculus for $\mathbf{ACT}$ is known. (An attempt was taken by Jipsen [12], but Buszkowski [5] showed that in Jipsen's system cut is not eliminable.)

Besides Kleene star, we also consider positive iteration, defined as $A^+ = A \cdot A^*$. One can prove in $\mathbf{ACT}$ (and therefore in $\mathbf{ACT}_\omega$), that A^* is equivalent to $\mathbf{1} \vee A^+$.

Positive iteration becomes important if we consider systems with Lambek's restriction, where antecedents of all sequents are required to be non-empty (as in the original paper by Lambek [21]). With Lambek's restriction, standard Kleene star becomes unavailable, and is replaced by positive iteration. Algebraically, this constraint corresponds to considering semigroups instead of monoids. Lambek's restriction is motivated by linguistic applications of the Lambek calculus and yields a system which is not a conservative fragment of the system without this restriction. Indeed, even some sequents with non-empty antecedents, like $(p \setminus p) \setminus q \to q$, require empty antecedents during the derivation. Thus, the study of systems with and without Lambek's restriction go in parallel. For more details, we refer to [20], and in this paper continue (for simplicity) using action logic without Lambek's restriction imposed.

Thanks to cut elimination, $\mathbf{ACT}_\omega$'s fragments with restricted sets of connectives are obtained by merely taking the appropriate subset of rules. In particular, removing rules for Kleene star yields $\mathbf{MALC}$ as a conservative fragment of $\mathbf{ACT}_\omega$. For $\mathbf{ACT}$, due to the lack of a cut-free sequent calculus, the question of conservative fragments is non-trivial. However, if a sequent $\Pi \to A$ does not include * and is derivable in $\mathbf{ACT}$, then it is derivable in $\mathbf{ACT}_\omega$, and therefore in $\mathbf{MALC}$ by conservativity of $\mathbf{ACT}_\omega$ over $\mathbf{MALC}$. Thus, $\mathbf{MALC}$ is a conservative fragment of both $\mathbf{ACT}$ and $\mathbf{ACT}_\omega$.

1.3 Complexity and Compactness Arguments

Kozen's completeness theorem [16] shows that for Kleene algebras, *i.e.,* in the language of $\cdot$, $\vee$, and *, though non-*-continuous Kleene algebras do exist [15], the logics for *-continuous Kleene algebras and all Kleene algebras coincide. Division operations change things dramatically. Namely, Buszkowski [5] shows that the derivability problem in $\mathbf{ACT}_\omega$ is Π^0_1-complete. On the other hand, **ACT** is clearly recursively enumerable. This has two corollaries:

(i) there exists an RKL which is not *-continuous;

(ii) there exists a sequent provable in $\mathbf{ACT}_\omega$, but not in **ACT** (in other words, true in all *-continuous RKL's, but not in all RKL's).

Buszkowski's argument, however, is not constructive, giving neither an example of a non-*-continuous RKL, nor a concrete sequent distinguishing $\mathbf{ACT}_\omega$ from **ACT**. We fill these gaps in Sections 2 and 3.

Before going further, let us mention other ways of proving statements (i) and (ii) above. The first statement can be proved by applying the well-known Gödel–Maltsev [10,23,24] compactness theorem [4]. Namely, one can write a first-order theory in the signature $\Omega = (\cdot, \backslash, /, \wedge, \vee, \mathbf{1}, {}^*; =, \preceq)$ whose models are exactly RKL's [5]. For example, Lambek's axiom for / becomes

$$\forall x \forall y \forall z\, (x \cdot y \preceq z \leftrightarrow x \preceq z \,/\, y);$$

for Kleene star one can take Pratt's equational axiomatisation (based on pure induction), *etc.* Notice that the strict version of the order, $x \prec y$, is expressible as $(x \preceq y) \,\&\, \neg(x = y)$. Now we add two new constant symbols, a and b, to the signature and the following axioms to the theory: $b \prec a^*$ and $a^n \prec b$ for each $n \geq 0$. Any finite subset of this theory includes only a finite number of such axioms and can be easily satisfied on an RKL. For example, one can take the powerset of the set of natural numbers with the set-theoretic lattice structure, $\cdot$ for elementwise addition, the Kleene star defined *-continuously, and $a = \{1\}$. Then take $b = \{0, 1, \dots, n_0, n_0 + 1\}$, where n_0 is the biggest value of n appearing in the axioms of the finite subset taken. By compactness theorem, the whole theory also has a model. This model is an RKL, but fails to be *-continuous: b is explicitly stated to be an upper bound for all a^n, which is smaller than a^*. This way of showing existence of non-*-continuous RKL's is, however, also not constructive.

As for (ii), there exists a way of making the complexity argument presented above in a sense constructive, using the notion of *productive function* [6]. Following the notation from Soare's book [34], let W_x be the x-th r.e. set (*i.e., x* is the natural number encoding the algorithm enumerating this set). A set P is called *productive*, if there exists a computable partial function ψ, such that if $W_x \subseteq P$, then $\psi(x)$ is defined and is an element of the set difference $P - W_x$. The function ψ itself is called *productive function.* The set $\overline{K} = \{x \mid x \notin W_x\}$ is clearly a productive one, with the trivial productive function $\psi_{\mathrm{id}}(x) = x$.

[4] Suggested by one of the anonymous reviewers.

[5] The $\wedge$ and $\vee$ *algebraic* operations here should not be confused with *logical* conjunction and disjunction used in first-order logic. For logical conjunction we use & instead of $\wedge$.

[6] Suggested by Fedor Pakhomov and Scott Weinstein.

Next, we use the following theorem [8, Theorem 2.1][26, Theorem 5][34, Theorem 4.5iii]: if P_1 is productive and it is m-reducible to P_2, then P_2 is also productive. Let P be the set of theorems of $\mathbf{ACT}_\omega$. By the Π_1^0-completeness result by Buszkowski [5], $\overline{K}$ is m-reducible to P. Therefore, P is a productive set with some computable productive function ψ. On the other hand, the set of theorems of **ACT** is r.e., *i.e.*, it is a W_y for some y; $W_y \subseteq P$, since **ACT** is weaker than $\mathbf{ACT}_\omega$. Thus, $\psi(y)$ is an element of P, but not W_y, in other words, gives an example of a sequent provable in $\mathbf{ACT}_\omega$, but not **ACT**.

Theoretically, this example can be explicitly extracted from the reasoning presented above. In order to do so, one needs to track the m-reduction of $\overline{K}$ to P used by Buszkowski (via the totality problem for context-free grammars) and transform it to a translation of the productive function from ψ_{id} to ψ. This yields a concrete algorithm for ψ, which can then be applied to y, which is the code of the algorithm for enumerating theorems of **ACT**; $\psi(y)$ is guaranteed to exist and it is the necessary example. In practice, however, performing this procedure is quite problematic. In Section 3 we give a much shorter example, which, moreover, exhibits some interesting structural features of proofs in action logic.

1.4 Models for ACT and $\mathbf{ACT}_\omega$

As mentioned by Buszkowski [5], standard Lindenbaum algebra construction shows that **ACT** and $\mathbf{ACT}_\omega$ are complete, respectively, w.r.t. the class of all RKL's and the class of *-continuous RKL's.

There are two more specific classes of RKL's which are usually considered as standard classes of models for the Lambek calculus and action logic.

The first one is the algebra $\mathcal{P}(\Sigma^*)$ of formal languages over a given alphabet Σ (here and further $\mathcal{P}(X)$ stands for the set of all subsets of X). The preorder is set-theoretical inclusion, and multiplication is defined as pairwise concatenation. It is well-known that this algebra is residuated. Language-theoretic division operations are defined as follows:

$$x \mathbin{/} y = \max\{z \mid z \cdot y \subseteq x\} = \{u \in \Sigma^* \mid (\forall v \in y)\, uv \in x\};$$

$$y \mathbin{\backslash} x = \max\{z \mid y \cdot z \subseteq x\} = \{u \in \Sigma^* \mid (\forall v \in y)\, vu \in x\}.$$

The preorder, $\subseteq$, enjoys arbitrary meets and joins and therefore induces a lattice structure. Finally, Kleene star is defined in the *-continuous way:

$$x^* = \sup\{x^n \mid n \geq 0\} = \{u_1 \dots u_n \mid n \geq 0, u_i \in x\}.$$

Models of the Lambek calculus and action logic on such algebras are called L-models.

The other class is formed by relational models, of the form $\mathcal{P}(W \times W)$, where W is a non-empty set. Elements of $\mathcal{P}(W \times W)$ are binary relations of W. Multiplication is defined as composition of relations. The lattice structure is set-theoretic. Relational algebras are also residuated. Kleene star x^* is defined as the reflexive-transitive closure of relation x. Models from this class are called R-models.

Both L- and R-models are *-continuous, so they form natural classes of models for $\mathbf{ACT}_\omega$. It is well-known, however, that there is an obstacle to completeness connected with distributivity. L- and R-models are distributive lattices, *i.e.,* enjoy $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. [7] On the other hand, one of the distributivity laws, namely $(A \vee B) \wedge (A \vee C) \to A \vee (B \wedge C)$, is not derivable in **MALC**, and therefore in $\mathbf{ACT}_\omega$. (The fact that distributive law is invalid in substructural logic was noticed by Ono and Komori [28].)

Another obstacle is connected with the unit constant, $\mathbf{1}$. Dividing the unit by something, $\mathbf{1} \mathbin{/} x$, yields a trivialisation of the interpretation and therefore extra sequents which are true, but not derivable. For both L- and R-models, such a sequent is, for example, $\mathbf{1} \mathbin{/} (p \mathbin{/} p) \to (\mathbf{1} \mathbin{/} (p \mathbin{/} p)) \cdot (\mathbf{1} \mathbin{/} (p \mathbin{/} p))$ [4,25,18].

These incompleteness issues actually have nothing to do with the Kleene star. In Section 4 we give a more fine-grained incompleteness result for language and relational semantics of $\mathbf{ACT}_\omega$, where the Kleene star plays a crucial role.

Despite incompleteness, the lower complexity bounds (Π^0_1-hardness), as shown by Buszkowski [4], are also valid for deciding general truth in R- and L-models. Essentially this comes from the fact that a fragment of $\mathbf{ACT}_\omega$ sufficient for encoding a Π^0_1-hard problem is indeed R- and L-complete (see footnote on page 508).

2 Example of an RKL which is not *-continuous

Kozen [15] presents an example of a Kleene algebra which is not *-continuous. In his construction, $\mathfrak{A} = \{\bot\} \cup (\mathbb{N} \times \mathbb{N}) \cup \{\top\}$, where $\mathbb{N} \times \mathbb{N}$ is ordered lexicographically, and $\bot$ and $\top$ are artificially added minimum and maximum. The following picture depicts the order on $\mathfrak{A}$:

$$\begin{array}{cccccc} \bullet & \longmapsto & \longmapsto & \longmapsto & \cdots & \bullet \\ \bot & \mathbb{N} & \mathbb{N} & \mathbb{N} & & \top \end{array}$$

Multiplication on $\mathbb{N}$ is componentwise addition: $(a_1, b_1) \cdot (a_2, b_2) = (a_1 + b_1, a_2 + b_2)$; $\bot \cdot x = x \cdot \bot = \bot$ and $\top \cdot y = y \cdot \top = \top$ for $y \neq \bot$. The Kleene star is defined as follows: $\bot^* = (0,0)^* = (0,0)$, which is the unit element, and for $x \succ (0,0)$ we have $x^* = \top$. One can see that $(0,1)^* = \top$, while $\sup\{(0,1)^n \mid n \geq 0\} = (1,0)$, which falsifies *-continuity.

This algebra has two extra properties: it is commutative and its order is linear.

Unfortunately, it is not residuated: for example, for any z of the form $(0,i)$ we have $z \cdot (1,1) = (1, i+1) \prec (2,0)$, but $(1,0) \cdot (1,1) = (2,1) \not\preceq (2,0)$. Thus, there is no $(2,0) \mathbin{/} (1,1) = \max\{z \mid z \cdot (1,1) \preceq (2,0)\}$.

This issue is quite a deep one, due to the following result which Restall [33] attributes to Pratt, calling it *Pratt's normality theorem* [33, Theorem 9.44]: in an RKL, if there exists $\sup\{a^n \mid n \geq 0\}$, then it coincides with a^*. In other words, whenever the *-continuous definition of Kleene star is available, we cannot choose another, non-standard version of Kleene star.

[7] As a basic fact of lattice theory, this also yields the dual: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

This gives us a key how to fix Kozen's construction and make it residuated: we should avoid suprema (least upper bounds) in our model. Let $\mathfrak{A} = \{\bot\} \cup (\{0\} \times \mathbb{N}) \cup (\{1, 2, \dots\} \times \mathbb{Z}) \cup \{\top\}$, with the following linear order:

$$\begin{array}{cccccc} \bullet & \longmapsto & \longleftrightarrow & \longleftrightarrow & \cdots & \bullet \\ \bot & \mathbb{N} & \mathbb{Z} & \mathbb{Z} & & \top \end{array}$$

Multiplication and Kleene star are defined exactly as in Kozen's model. The same reasoning shows that Kleene star is not *-continuous; $\top$ is still the smallest fixpoint, and the supremum merely does not exist, since $\mathbb{Z}$ has no minimal element. The only thing we need to check is that division (due to commutativity, we have only one division) here is correctly defined:

Lemma 2.1 *For any $x, y \in \mathfrak{A}$, there exists* $\max\{z \mid z \cdot y \preceq x\}$ *(which we denote by $x \,/\, y = y \setminus x$).*

Proof. Easily, $x \,/\, \bot = \top$, $\top \,/\, y = \top$, $\bot \,/\, y = \bot$ (for $y \neq \bot$), $x \,/\, \top = \bot$ (for $x \neq \top$).

Now let x and y be pairs of numbers, $x = (a_1, a_2)$, $y = (b_1, b_2)$. Let us call the fragment $\{i\} \times \mathbb{Z}$ ($i > 0$) the i-th galaxy of $\mathfrak{A}$; the 0-th galaxy is truncated, $\{0\} \times \mathbb{N}$. There are four possible cases:

- $a_1 < b_1$. Then $b_1 - a_1 > 0$, and therefore the $(b_1 - a_1)$-th galaxy is not truncated, the second component there allows unrestricted subtraction, and we have $x \,/\, y = (b_1 - a_1, b_2 - a_2)$.
- $a_1 = b_1$, $a_2 \leq b_2$. Then $x \,/\, y = (0, b_2 - a_2)$.
- $a_1 = b_1$, $a_2 > b_2$. Then $x \,/\, y = \bot$, since for no pair $(c_1, c_2) \in \mathfrak{A}$ we can have $(a_1 + c_1, a_2 + c_2) \preceq (b_1, b_2)$.
- $a_1 > b_1$. Then $x \,/\, y = \bot$.

$\square$

This example of non-*-continuous RKL inherits two extra properties of Kozen's example: commutativity and linearity of the order.

3 Example of a Sequent Distinguishing ACT and $\mathbf{ACT}_\omega$

In this section we present a concrete example of a sequent provable in $\mathbf{ACT}_\omega$, but not in **ACT**. This is obtained by presenting a new induction principle for action logic, which is admissible in the *-continuous situation, but does not follow from the fixpoint axiomatisation of Kleene star (Pratt's pure induction principle). We call it the "induction-in-the-middle" rule:

Lemma 3.1 *The following rule*

$$\frac{\to B \quad A \to B \quad A, B, A \to B}{A^* \to B} \; ({}^* \to)_{\mathrm{mid}}$$

is admissible in $\mathbf{ACT}_\omega$.

Proof. By induction on n we show that all $A^n \to B$ are derivable from the premises using cut. Then apply the ω-rule. $\square$

The next theorem shows, however, that $({}^* \to)_{\mathrm{mid}}$ is not valid in **ACT**. This gives a concrete example of a sequent that distinguishes **ACT** and $\mathbf{ACT}_\omega$.

Theorem 3.2 *The sequent*

$$(p \wedge q \wedge (p \,/\, q) \wedge (p \,\backslash\, q))^+ \to p$$

is provable in $\mathbf{ACT} + ({}^* \to)_{\mathrm{mid}}$ *(and therefore in* $\mathbf{ACT}_\omega$*), but not in* **ACT**.

Proof. Proving $(p \wedge q \wedge (p \,/\, q) \wedge (p \,\backslash\, q))^+ \to p$ in $\mathbf{ACT} + ({}^* \to)_{\mathrm{mid}}$ is easy. First we establish a variant of the induction-in-the-middle rule for ${}^+$:

$$\dfrac{A \to B \quad A^2 \to B \quad A, B, A \to B}{A^+ \to B}\ ({}^+ \to)_{\mathrm{mid}}$$

This new rule is obtained from $({}^* \to)_{\mathrm{mid}}$ by the following derivation (recall that $A^+ = A \cdot A^*$):

$$\dfrac{\dfrac{\dfrac{A \to B}{\to A \,\backslash\, B}\,(\to \backslash) \quad \dfrac{A^2 \to B}{A \to A \,\backslash\, B}\,(\to \backslash) \quad \dfrac{\dfrac{A \to A \quad A, B, A \to B}{A, A, A \,\backslash\, B, A \to B}\,(\backslash \to)}{A, A \,\backslash\, B, A \to A \,\backslash\, B}\,(\to \backslash)}{A^* \to A \,\backslash\, B}\,({}^* \to)_{\mathrm{mid}} \quad \dfrac{A \to A \quad B \to B}{A, A \,\backslash\, B \to B}\,(\backslash \to)}{\dfrac{A, A^* \to B}{A \cdot A^* \to B}\,(\cdot \to)}\,(\mathrm{cut})$$

Next, since $p \to p$, and $p \,/\, q, q \to p$, and $p \,/\, q, p, p \,\backslash\, q \to p$ are derivable in the Lambek calculus, for $A = (p \wedge q \wedge (p \,/\, q) \wedge (p \,\backslash\, q))$ we have $A \to p$, and $A^2 \to p$, and $A, p, A \to p$, by applying $(\wedge \to)$ several times. Now the $({}^+ \to)_{\mathrm{mid}}$ rule yields the necessary sequent $A^+ \to p$.

In order to show that this sequent is not derivable in **ACT**, we construct a counter-model, *i.e.,* an RKL in which this sequent is false. [8]

Fix a two-letter alphabet $\Sigma = \{a, c\}$ and consider two families of formal languages, $\mathfrak{L}_1$ and $\mathfrak{L}_2$, defined as follows:

(i) $\mathfrak{L}_1$ is the family of all finite subsets of $\{a^n \mid n \geq 0\}$, including $\varnothing$ and $\{\varepsilon\}$ (ε is the empty word), which will be the zero and the unit of our RKL;

(ii) $\mathfrak{L}_2$ is the family of all languages of the form

$$A \cup \bigcup_{h \geq 0} \{a^i c a^{i+h} \mid i \geq f_{\mathrm{R}}(h)\} \cup \bigcup_{h > 0} \{a^{i+h} c a^i \mid i \geq f_{\mathrm{L}}(h)\},$$

where A is a cofinite subset of $\{a^n \mid n \geq 0\}$ and f_{L}, $f_{\mathrm{R}} \colon \mathbb{N} \to \mathbb{N}$ are functions of at least linear growth. Throughout this paper, "f is a function of at least linear growth" means that, for all h, $f(h) \geq \alpha h + \beta$ for some rational α and β, $\alpha > 0$.

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[8] We cannot use the non-*-continuous RKL from Section 2 here, since it is commutative, and in the commutative case $({}^* \to)_{\mathrm{mid}}$ becomes admissible in **ACT**.

Let $\mathfrak{L} = \mathfrak{L}_1 \cup \mathfrak{L}_2$ and ∞ be an extra object (fresh constant) not belonging to $\mathfrak{L}$. Now let us define an RKL structure on $\mathfrak{A} = \mathfrak{L} \cup \{\infty\}$.

(i) The preorder, $\preceq$, is defined as set-theoretic inclusion, $\subseteq$, on $\mathfrak{L}$, and ∞ is declared the maximal element. Since $\mathfrak{L}$ is closed under (finite) unions and intersections (for $x, y \in \mathfrak{L}_2$ it follows from the fact that pointwise minimum and maximum of two functions of at least linear growth are again functions of at least linear growth; other cases are obvious), this preorder forms a lattice structure on $\mathfrak{A}$.

(ii) Multiplication, $x \cdot y$, is defined as follows:
- pairwise concatenation (as in L-models), if x and y are both in $\mathfrak{L}_1$ or if one of the arguments is in $\mathfrak{L}_1$ and the other is in $\mathfrak{L}_2$ (for the second case correctness follows from Lemma 3.3 below);
- ∞, if x and y are both in $\mathfrak{L}_2$;
- $\infty \cdot \varnothing = \varnothing \cdot \infty = \varnothing$;
- $\infty \cdot x = x \cdot \infty = \infty$, if $x \neq \varnothing$.

Checking associativity is routine. The unit is $\{\varepsilon\}$.

(iii) Left and right divisions are correctly defined due to Lemma 3.4 below.

(iv) Finally, Kleene star is defined as follows: $\varnothing^* = \{\varepsilon\}^* = \{\varepsilon\}$; $x^* = \infty$ for $x \neq \varnothing, \{\varepsilon\}$. We show that this definition is correct by verifying Pratt's pure induction condition, $(x \,/\, x)^* = x \,/\, x$. For $x = \varnothing$ and $x = \infty$ we have $x \,/\, x = \infty$, $\infty^* = \infty$. For $x \neq \varnothing, \infty$ we use Lemma 3.5 below showing that $x \,/\, x = \{\varepsilon\}$; $\{\varepsilon\}^* = \{\varepsilon\}$. Other conditions accompanying pure induction in Pratt's system are obvious: our definition of Kleene star is clearly monotone, for any x we have $\mathbf{1} \preceq x^*$ and $x \preceq x^*$, and since x^* is either $\{\varepsilon\}$ or ∞, we also have $x^* \cdot x^* \preceq x^*$.

Below we state and prove several lemmata supporting correctness of this definition.

Lemma 3.3 *If $x \in \mathfrak{L}_1$, $x \neq \varnothing$, and $y \in \mathfrak{L}_2$, then $x \cdot y$ and $y \cdot x$, where multiplication is defined as pairwise concatenation, are both in $\mathfrak{L}_2$. (If $x = \varnothing$, then $x \cdot y = y \cdot x = \varnothing \in \mathfrak{L}_1$.)*

Proof. We show that $x \cdot y \in \mathfrak{L}_2$ (the statement for $y \cdot x$ is symmetric). Since finite unions of languages from $\mathfrak{L}_2$ belong to $\mathfrak{L}_2$, it is sufficient to consider $x = \{a^k\}$ and show that $\{a^k\} \cdot y \in \mathfrak{L}_2$. Recall that

$$y = A \cup \bigcup_{h \geq 0} \{a^i c a^{i+h} \mid i \geq f_{\mathrm{R}}(h)\} \cup \bigcup_{h > 0} \{a^{i+h} c a^i \mid i \geq f_{\mathrm{L}}(h)\},$$

whence

$$\{a^k\} \cdot y = \{a^k\} \cdot A \cup \bigcup_{h \geq 0} \{a^{i+k} c a^{i+h} \mid i \geq f_{\mathrm{R}}(h)\} \cup \bigcup_{h > 0} \{a^{i+h+k} c a^i \mid i \geq f_{\mathrm{L}}(h)\}.$$

Apply the following transformations:

$$\bigcup_{h > 0} \{a^{i+h+k} c a^i \mid i \geq f_{\mathrm{L}}(h)\} = \bigcup_{\ell_1 > k} \{a^{i+\ell_1} c a^i \mid i \geq f_{\mathrm{L}}(\ell_1 - k)\},$$

where $\ell_1 = h + k$;

$$\bigcup_{h \geq k} \{a^{i+k} c a^{i+h} \mid i \geq f_{\mathrm{R}}(h)\} = \bigcup_{\ell_2 \geq 0} \{a^{j_2} c a^{j_2 + \ell_2} \mid j_2 \geq f_{\mathrm{R}}(\ell_2 + k) + k\},$$

where $\ell_2 = h - k$, $j_2 = i + k$;

$$\bigcup_{0 \leq h < k} \{a^{i+k} c a^{i+h} \mid i \geq f_{\mathrm{R}}(h)\} = \bigcup_{0 < \ell_3 \leq k} \{a^{j_3 + \ell_3} c a^{j_3} \mid f_{\mathrm{R}}(k - \ell_3) + k - \ell_3\},$$

where $\ell_3 = k - h$, $j_3 = i + h$.

This yields

$$\{a^k\} \cdot y = \widetilde{A} \cup \bigcup_{\ell \geq 0} \{a^i c a^{i+\ell} \mid i \geq \tilde{f}_{\mathrm{R}}(\ell)\} \cup \bigcup_{\ell > 0} \{a^{i+\ell} c a^i \mid i \geq \tilde{f}_{\mathrm{L}}(\ell)\},$$

where $\widetilde{A} = \{a^k\} \cdot A$ is a cofinite subset of $\{a^n \mid n \geq 0\}$ and $\tilde{f}_{\mathrm{R}}$ and $\tilde{f}_{\mathrm{L}}$, defined as follows

$$\tilde{f}_{\mathrm{R}}(\ell) = f_{\mathrm{R}}(\ell + k) + k;$$
$$\tilde{f}_{\mathrm{L}}(\ell) = \begin{cases} f_{\mathrm{L}}(\ell - k) & \text{for } \ell > k, \\ f_{\mathrm{R}}(k - \ell) + k - \ell & \text{for } 0 < \ell \leq k, \end{cases}$$

are both functions of at least linear growth [9] ($\tilde{f}_{\mathrm{L}}$ can actually decrease on $h \leq k$, but at least linear growth is an asymptotic property). Therefore $\{a^k\} \cdot y \in \mathfrak{L}_2$.$\square$

Lemma 3.4 *For any elements $x, y \in \mathfrak{A}$ there exist $x \,/\, y = \max\{z \mid z \cdot y \preceq x\}$ and $y \setminus x = \max\{z \mid y \cdot z \preceq x\}$.*

Proof. Consider only $x \,/\, y$ ($y \setminus x$ is symmetric). First we handle some degenerate cases:

- $y = \varnothing$: since $z \cdot \varnothing = \varnothing$ for any z, we have $x \,/\, \varnothing = \infty$. In particular, $\varnothing \,/\, \varnothing = \infty$.
- $x = \infty$: since $z \cdot y \preceq \infty$ for any z, we have $\infty \,/\, y = \infty$. In particular, $\infty \,/\, \infty = \infty$.
- $y = \infty$, $x \neq \infty$: since $z \cdot \infty \preceq x$ holds only for $z = \varnothing$ (in this case we get $\varnothing \preceq x$, otherwise $\infty \not\preceq x$), we have $x \,/\, \infty = \varnothing$.

Thus, now we have only the interesting case of $x, y \in \mathfrak{L}$, $y \neq \varnothing$. First we show that in this case $x \,/\, y$ can be defined exactly as in L-models (Subsection 1.4). Namely, if the language $z_0 = \{u \in \{a, c\}^* \mid (\forall v \in y)\, uv \in x\}$ (the language-theoretic division of x by y) belongs to $\mathfrak{L}$, then $z_0 = x \,/\, y = \max\{z \cdot y \preceq x\}$ in $\mathfrak{A}$ (*i.e.,* maximum is taken over all elements of $\mathfrak{A}$, including ∞, and w.r.t. $\preceq$).

Indeed, $z_0 \cdot y \subseteq x$, and therefore $z_0 \cdot y \preceq x$. Now let $z \cdot y \preceq x$ for some other z. Since $y \neq \varnothing$, z cannot be ∞ ($\infty \cdot y = \infty \not\preceq x$). Hence, $z \in \mathfrak{L}$, $z \cdot y \subseteq x$, and

[9] Recall that k is constant.

since z_0 is the language-theoretic division, $z \subseteq z_0$ and therefore $z \preceq z_0$. Thus, z_0 is the maximum in $\mathfrak{A}$.

Now it is sufficient to show that $\mathfrak{L}$ is closed under language-theoretic division operations (with non-zero denominator). Consider four possible cases:

- $x, y \in \mathfrak{L}_1$. The class of finite language over an alphabet is closed under language-theoretic division, provided the denominator is not $\varnothing$.
- $x \in \mathfrak{L}_1$, $y \in \mathfrak{L}_2$. In this case $x \,/\, y = \varnothing$, since there exists a word $v_0 = a^{i_0} c a^{i_0} \in y$, and for any $u \in \{a, c\}^*$ the word uv_0 contains c, and therefore cannot belong to x.
- $x \in \mathfrak{L}_2$, $y \in \mathfrak{L}_1$. Since in L-models $x \,/\, (y_1 \cup \ldots \cup y_n) = (x \,/\, y_1) \cap \ldots \cap (x \,/\, y_n)$, and y, being a non-empty finite set, is a finite union of singletons, it is sufficient to show that $z_0 = x \,/\, \{a^k\} = \{u \in \{a, c\}^* \mid ua^k \in x\}$ belongs to $\mathfrak{L}_2$. Recall that x, being a language from $\mathfrak{L}_2$, has the form

$$x = A \cup \bigcup_{h \geq 0} \{a^i c a^{i+h} \mid i \geq f_{\mathrm{R}}(h)\} \cup \bigcup_{h > 0} \{a^{i+h} c a^i \mid i \geq f_{\mathrm{L}}(h)\}.$$

Next, since in L-models for $x = \bigcup_\gamma x_\gamma$ (x is an infinite union) we have $x \,/\, \{a^k\} = \bigcup_\gamma (x_\gamma \,/\, \{a^k\})$, we can divide each component of x by $\{a^k\}$ independently:

$$\bigcup_{h \geq k} \{a^i c a^{i+h} \mid i \geq f_{\mathrm{R}}(h)\} \,/\, \{a^k\} = \bigcup_{\ell_1 \geq 0} \{a^i c a^{i+\ell_1} \mid i \geq f_{\mathrm{R}}(\ell_1 + k)\},$$

where $\ell_1 = h - k$;

$$\bigcup_{0 \leq h < k} \{a^i c a^{i+h} \mid i \geq f_{\mathrm{R}}(h)\} \,/\, \{a^k\} = \bigcup_{0 < \ell_2 \leq k} \{a^{j_2 + \ell_2} c a^{j_2} \mid j_2 \geq \max\{0, f_{\mathrm{R}}(k - \ell_2) - \ell_2\}\},$$

where $\ell_2 = k - h, j_2 = i + h - k = i - \ell_2$;

$$\bigcup_{h > 0} \{a^{i+h} c a^i \mid i \geq f_{\mathrm{L}}(h)\} \,/\, \{a^k\} = \bigcup_{\ell_3 > k} \{a^{j_3 + \ell_3} c a^{j_3} \mid j_3 \geq \max\{0, f_{\mathrm{L}}(\ell_3 - k) - k\}\},$$

where $\ell_3 = h + k$, $j_3 = i - k$.

Thus,

$$x \,/\, \{a^k\} = \widetilde{A} \cup \bigcup_{h \geq 0} \{a^i c a^{i+h} \mid i \geq \tilde{f}_{\mathrm{R}}(h)\} \cup \bigcup_{h > 0} \{a^{i+h} c a^i \mid i \geq \tilde{f}_{\mathrm{L}}(h)\},$$

where $\widetilde{A} = A \,/\, \{a^k\}$ is a cofinite subset of $\{a^n \mid n \geq 0\}$, and $\tilde{f}_{\mathrm{R}}$ and $\tilde{f}_{\mathrm{L}}$, defined as follows

$$\tilde{f}_{\mathrm{R}}(\ell) = f_{\mathrm{R}}(\ell + k);$$
$$\tilde{f}_{\mathrm{L}}(\ell) = \begin{cases} \max\{0, f_{\mathrm{R}}(k-\ell) - \ell\} & \text{for } 0 < h \le k, \\ \max\{0, f_{\mathrm{L}}(\ell - k) - k\} & \text{for } h > k, \end{cases}$$

are functions of linear growth (again, for $\tilde{f}_{\mathrm{L}}$ growth starts from $h > k$).

- $x, y \in \mathfrak{L}_2$. We show that in this case $z_0 = \{u \in \{a,c\}^* \mid (\forall v \in y)\, uv \in x\}$ belongs to $\mathfrak{L}_1$. Indeed, if u includes at least one letter c, then for $v_0 = a^{i_0} c a^{i_0} \in y$ the word uv_0 includes at least two c's, and cannot belong to y. Thus, $z_0 \subseteq \{a^n \mid n \ge 0\}$, and it remains to show that z_0 is finite. Let a^k be an element of z_0. Fix an arbitrary $a^i c a^i \in y$ (such an element exists by definition of $\mathfrak{L}_2$) and multiply it by a^k. We get $a^i c a^{i+k} \in x$. This yields $i \ge f_{\mathrm{R}}(k)$, where f_{R} is taken from the $\mathfrak{L}_2$ representation of x, and by growth condition $i \ge \alpha k + \beta$. Since $\alpha > 0$, we get $k \le (i-\beta)/\alpha$, which establishes a global boundary for possible values of k: recall that i, α, and β were taken independently from k. Therefore, $z_0 \subseteq \{a^k \mid 0 \le k \le (i-\beta)/\alpha\}$ is finite and belongs to $\mathfrak{L}_1$.

$\square$

Lemma 3.5 *For any x, except $\varnothing$ and ∞, $x \,/\, x = \{\varepsilon\}$.*

Proof. Since $\{\varepsilon\}$ is the unit of $\mathfrak{A}$, $\{\varepsilon\} \cdot x = x$, therefore $x \,/\, x \succeq \{\varepsilon\}$. Suppose there exists $z \succ \{\varepsilon\}$ such that $z \cdot x \preceq x$. Then $\{a^k\} \preceq z$ for some $k > 0$, and by monotonicity $\{a^k\} \cdot x \preceq x$ (monotonicity of $\cdot$ w.r.t. $\preceq$ holds in all residuated lattices, since the corresponding logical rule is admissible in the Lambek calculus [21]). Consider two cases:

- $x \in \mathfrak{L}_1$, $x \ne \varnothing$. Then let a^m be the element of x with the greatest m. Clearly, $a^k a^m \notin x$, therefore $\{a^k\} \cdot x \not\preceq x$.
- $x \in \mathfrak{L}_2$. Take $a^{i_0} c a^{i_0} \in x$ (exists by definition of $\mathfrak{L}_2$). Since $\{a^k\} \cdot x \preceq x$, we have $a^{i_0+k} c a^{i_0} \in x$, $a^{i_0+2k} c a^{i_0} \in x$, $\ldots$, $a^{i_0+mk} c a^{i_0} \in x$, $\ldots$ Thus, $f_{\mathrm{L}}(mk) \le i_0$ for arbitrary big m, which contradicts with the growth condition for f_{L}.

$\square$

Now we finish the proof of Theorem 3.2 by falsifying $(p \wedge q \wedge (p \,/\, q) \wedge (p \setminus q))^+ \to p$ in the newly constructed RKL:

Lemma 3.6 *If p is interpreted as the language from $\mathfrak{L}_2$ with $A = \{a^n \mid n \ge 0\}$ and $f_{\mathrm{L}}(h) = f_{\mathrm{R}}(h) = 2h$, and $q = p \cdot \{a\}$, then $p \wedge q \wedge (p \,/\, q) \wedge (p \setminus q) = \{a\}$. Thus, since $\{a\}^+ = \infty \not\preceq p$, the sequent $(p \wedge q \wedge (p \,/\, q) \wedge (p \setminus q))^+ \to p$ is not true under this intepretation.*

Proof. Clearly, $a \in p$, and, since $\varepsilon \in p$, also $a \in q$. Next, $p \cdot \{a\} \preceq q$ yields $\{a\} \preceq p \setminus q$.

Next, let us show that $p \,/\, q = \{a\}$. As follows from the proof of Lemma 3.4, since both p and q are elements of $\mathfrak{L}_2$, $p \,/\, q \in \mathfrak{L}_1$, *i.e.,* it is a finite subset of

$\{a^n \mid n \geq 0\}$. This means that $p \,/\, q = \{a^n \mid (\forall v \in q)\, a^n v \in p\}$, and since any $v \in q$ is of the form ua, where $u \in p$, we have $p \,/\, q = \{a^n \mid (\forall u \in p)\, a^n ua \in p\}$.

For $n = 1$ we indeed have $aua \in p$ for any $u \in p$ (all three components in the $\mathfrak{L}_2$ representation of p are upwardly closed under multiplication by a on both sides).

For $n = 0$, take $u = c$. Since $f_{\mathrm{R}}(0) = 0$, it is an element of p. The word $ua = ca$, however, is of the form $a^i ca^{i+h}$ for $i = 0$ and $h = 1$, and since $f_{\mathrm{R}}(1) = 2 > 0$, this word does not belong to p. Therefore, $\varepsilon = a^0 \notin p \,/\, q$.

Analogous reasoning applies to $n > 1$. Again, take $u = c \in p$ and consider the word $a^n ua = a^n ca$, which is of the form $a^{i+h} ca^i$ for $i = 1$ and $h = n-1 > 0$. Having $f_{\mathrm{L}}(n-1) = 2(n-1) \geq 2 > 1$, we show that this word is not in p, therefore $a^n \notin p \,/\, q$ for $n > 1$.

Finally, having $p \,/\, q = \{a\}$ and p, q, $p \setminus q \succeq \{a\}$, we obtain $p \wedge q \wedge (p \,/\, q) \wedge (p \setminus q) = \{a\}$. □

This finishes the proof of Theorem 3.2. □

We conclude this section with some general remarks. The search for concrete formulae witnessing the fact that ω-rules are more powerful than induction has a long history, starting from Gödel's Second Incompleteness Theorem [11]. The formula there is the well-known consistency statement: "for all n, n does not encode a proof of contradiction." Later, other arithmetical statements that are true, but not provable in Peano arithmetic (*i.e.,* using the standard induction principle), of more combinatorial nature, were discovered. These include Hercules vs. Hydra by Kirby and Paris [13], Beklemishev's worm [2], *etc.* An example closer to our discussion was discovered by Kozen for PDL [14]. The key feature of our example is that it is formulated in a propositional language, while the formulae mentioned above are first-order ones. On the other hand, our example actually shows not the weakness of induction in general, but rather the fact that in action logic there exist induction principles different from (and not following from) Pratt's pure induction, but yet not transfinite.

4 Incompleteness of $\mathbf{ACT}_\omega(\cdot, \setminus, /, \wedge, {}^*)$ w.r.t. R- and L-models

Though relational and language models, both being *-continuous, are natural classes of interpretations for $\mathbf{ACT}_\omega$, there are well-known obstacles to completeness connected with distributivity and the unit constant (see Subsection 1.4). Without these problematic connectives, $\vee$ and $\mathbf{1}$, there is a hope for R- and L-completeness. This hope is supported by completeness results for the corresponding fragment without *, $\mathbf{MALC}(\cdot, \setminus, /, \wedge)$: see Andréka and Mikulás [1] for R-models of $\mathbf{MALC}(\cdot, \setminus, /, \wedge)$, Buszkowski [3] for L-models of the product-free system $\mathbf{MALC}(\setminus, /, \wedge)$, Pentus [30,31] for L-models of the Lambek calculus (in our notation, $\mathbf{MALC}(\cdot, \setminus, /)$). L-completeness of $\mathbf{MALC}(\cdot, \setminus, /, \wedge)$, with both multiplication and intersection, is still an open problem, and there are no arguments against the positive answer.

This motivates us to consider the fragment $\mathbf{ACT}_\omega(\cdot, \setminus, /, \wedge, {}^*)$ and conjecture its R- and L-completeness. Disjunction, however, is hidden inside Kleene star: $a^* = \mathbf{1} \vee a^+$, and distributivity can shoot around the corner, making $\mathbf{ACT}_\omega$ incomplete even in this restricted fragment.

Theorem 4.1 *The sequent*

$$(s \,/\, (r \,/\, r)) \wedge (s \,/\, (p^+ \wedge q^+)) \to s \,/\, (p^* \wedge q^*)$$

is true in all distributive RKL's, but not provable in $\mathbf{ACT}_\omega$.

Proof. First we show that this sequent is true in all distributive RKL's. Recall that $A^* \leftrightarrow \mathbf{1} \vee A^+$. By distributivity and monotonicity of $\wedge$, we get

$$p^* \wedge q^* \to (\mathbf{1} \vee p^+) \wedge (\mathbf{1} \vee q^+) \to \mathbf{1} \vee (p^+ \wedge q^+) \to (r \,/\, r) \vee (p^+ \wedge q^+).$$

Next, we put this under $s \,/ \ldots$ (the direction of the arrow changes), which allows us to replace $\vee$ by $\wedge$ using $(A \,/\, B) \wedge (A \,/\, C) \to A \,/\, (B \vee C)$:

$$(s \,/\, (r \,/\, r)) \wedge (s \,/\, (p^+ \wedge q^+)) \to s \,/\, ((r \,/\, r) \vee (p^+ \wedge q^+)) \to s \,/\, (p^* \wedge q^*).$$

The reasoning above can be done in $\mathbf{ACT}$ + distributivity axiom. Thus, the goal sequent is true in all distributive RKL's.

For the second part, non-derivability in $\mathbf{ACT}_\omega$, we recall that cut is eliminable in this calculus [29] and perform exhaustive proof search. Since the $(\to /)$ rule is invertible, we can suppose that it was applied as the last step of the derivation:

$$\frac{(s \,/\, (r \,/\, r)) \wedge (s \,/\, (p^+ \wedge q^+)), p^* \wedge q^* \to s}{(s \,/\, (r \,/\, r)) \wedge (s \,/\, (p^+ \wedge q^+)) \to s \,/\, (p^* \wedge q^*)} \ (\to /)$$

Now we have four options for applying the $(\wedge \to)$ rule, with the following premises:

(i) $s \,/\, (r \,/\, r), p^* \wedge q^* \to s$;

(ii) $s \,/\, (p^+ \wedge q^+), p^* \wedge q^* \to s$;

(iii) $(s \,/\, (r \,/\, r)) \wedge (s \,/\, (p^+ \wedge q^+)), p^* \to s$;

(iv) $(s \,/\, (r \,/\, r)) \wedge (s \,/\, (p^+ \wedge q^+)), q^* \to s$.

These sequents are not generally true in L-models, and therefore are not derivable in $\mathbf{ACT}_\omega$. The counter-interpretations are as follows:

(i) $s = r = \{\varepsilon\}$, $p = q = \{a\}$;

(ii) $s = \{a\}^+$, $p = q = \{a\}$;

(iii) $s = r = \{\varepsilon\}$, $p = \{a\}$, $q = \varnothing$;

(iv) $s = r = \{\varepsilon\}$, $p = \varnothing$, $q = \{a\}$.

$\square$

5 Conclusions and Future Work

The counter-model used in the proof of Theorem 3.2 is quite an *ad hoc* invention to falsify one particular sequent. On the other hand, it is very closely related to L-models (which are all *-continuous, and therefore useless in connection to Theorem 3.2). It looks interesting to perform a more systematic study of variants of L-models with Kleene star, and possibly find natural classes of models which are not *-continuous (*i.e.,* give semantics for **ACT**, but not $\mathbf{ACT}_\omega$). One possible starting point for such a study is the interpretation of **MALC** on syntactic concept lattices (SCL's) by Wurm [35]. SCL's are much like L-models, but can be, for example, non-distributive, thus avoiding incompleteness issues for meet and join.

Our $({}^* \to)_{\mathrm{mid}}$ rule is actually one of an infinite series of induction-in-the-middle rules arising from circular proof systems [20]:

$$\frac{\to B \quad A \to B \quad \dots \quad A^{m+k-1} \to B \quad A^m, B, A^k \to B}{A^* \to B}$$

All these rules are admissible in $\mathbf{ACT}_\omega$, but even if we add all of them, we still get a system weaker than $\mathbf{ACT}_\omega$ (due to complexity reasons: it is still r.e.). The question is how do these rules interact with each other: for example, is there a finite subfamily of these rules which derives all of them?

Our example of a sequent derivable in $\mathbf{ACT}_\omega$, but not in **ACT**, contains additive conjunction, $\wedge$. Buszkowski [5], however, also proves Π_1^0-completeness for the fragment with $\vee$ instead of $\wedge$. Moreover, in the view of the Π_1^0-completeness result [20] for a closely related system, with Lambek's restriction, it is highly likely that even in the multiplicative-only fragment, including $\cdot$, $\backslash$, $/$, and *, the infinitary system $\mathbf{ACT}_\omega$ is stronger than **ACT**. The task of finding concrete examples of sequents to distinguish the two systems in these restricted fragments is left open for future research.

As for L- and R-completeness, the question is still open whether the purely multiplicative Lambek calculus with iteration (in our notation, $\mathbf{ACT}_\omega(\cdot, \backslash, /, {}^*)$) is L-complete (R-complete). [10]

Finally, we still have the old problem of constructing a good (cut-free) Gentzen-style system for **ACT**. For Kleene algebra, there is a recent approach by Das and Pous [7], who present a cut-free hypersequential calculus with circular proofs. Unfortunately, their approach cannot be directly generalised to the residuated case.

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[10] L-completeness is known for two fragments of this calculus. In the first fragment, Kleene star is allowed only on the top level in formulae of the antecedent. In this case completeness follows from that of the original Lambek calculus due to invertibility of $({}^* \to)$. The second fragment is the product-free Lambek calculus with * allowed only in denominators of $\backslash$ and $/$ (*i.e.,* in subformulae of the form $A^* \setminus B$ and $B \,/\, A^*$). L-completeness for the variant of this fragment, with Lambek's restriction and positive iteration in place of Kleene star, is stated in [19].

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