Cut-Free Sequent Calculi for Products and Relativised Products of Modal Logics

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Abstract

Using structured labels, we define one-sided, contraction-free and cut-free sequent calculi for a class of products and relativised products of modal logics, including some examples which can not be characterised by a finite Hilbert-style axiomatisation. To this end, we introduce the product of labelled calculi. For the special case of products with S5, the method yields two different systems, one based on equivalence frames the other on universal frames, and we show how to translate derivations from one to the other and vice versa. In these calculi, all logical rules are height-preserving invertible. Furthermore, we prove that contraction and cut are admissible.

Keywords: proof theory, cut-free sequent calculus, labelled deduction, products of modal logics, relativised product.

1 Introduction

Building products of structures is a very natural standard construction in mathematics. The product formation of Kripke structures is just a special case. The *product* of n modal *logics* $\mathcal{L}_1, \ldots, \mathcal{L}_n$ is defined to be the set of modal formulas that hold in all products of frames $\mathcal{F}_1, \ldots, \mathcal{F}_n$ where each $\mathcal{F}_i$ is a frame for $\mathcal{L}_i$. As a combination method for modal logics, products have been introduced in [17,18,3]. *Relativised products* are determined by classes of subframes of product frames. A detailed discussion of these notions and major results can be found in [2] and [8]. In particular, axiomatisations for interesting classes are known but there are also very natural logics as the n-dimensional products $\mathbf{K}^n$ and $\mathbf{S5}^n$, $n \geq 3$, which are not finitely axiomatisable (see [7] for $\mathbf{K}^n$, [6,9] for further results).

Labelled calculi provide a general method for constructing *sequent systems* for modal logics, see [11,4]. In that approach, the basic judgements are labelled formulas $x\colon A$ or relational atoms xRy where x, y are labels that denote worlds in a Kripke frame. The statement $x : A$ could be read as "A holds at world x", the expression xRy stands for "y is reachable from x". The different systems

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share a set of logical rules, while each of them includes a set of *frame rules* that reflect the special conditions for the Kripke frames in question.

Since then, several extensions of the language of judgements have been considered. To capture logics based on neighbourhood semantics, labelled calculi have been defined in [13,5,12] that use a multi-sorted labelled language and judgements $a \Vdash^{\exists} A$ and $a \Vdash^{\forall} A$ where the first assertion means that formula A holds at some world x in a and the second stands for "A holds at all worlds x in a". For the logic of subset spaces, a labelled sequent calculus has been presented in [1]. One characteristic of subset spaces is that formulas are evaluated with respect to pairs of *points* x and *opens* u so that $x \in u$, and this is used in the design of the calculus. The language uses two sorts of labels, labelled formulas have the form $(x, u) : A$ where x, u are labels of sort 1 or 2 respectively, and a third type of judgement is introduced referring to the status of the pair (x, u) as a world. The ingredients of the latter approach are used and generalised here to deal with products and relativised products.

In order to obtain *calculi for products*, we define the *product of calculi.* Throughout we work with one-sided sequent systems in the style of the GS-calculi in [16], but extended by labels. The calculi to be combined are cut-free unimodal systems with pairwise distinct modalities. Usually they are formulated with a corresponding symbol for the accessibility relation but we allow the exception of the extremely simple system for S5 which is based on universal frames. The language of the compound system contains the modalities and relational symbols of the components plus a world-predicate. The labels attached to formulas are tuples of simple labels. The logical rules are adapted to this situation and extra frame rules can be added to determine the type of relativisation.

Arbitrary relativised products, expanding products, and non-relativised products serve as examples for instantiations of the general scheme. For the non-relativised case, an equivalent simplified variant is also presented. To illustrate how to work with the calculi, we consider the alternative ways to describe an S5-component, either based on equivalence frames or based on universal frames. In our setting, the equivalence of the two approaches takes the form of comparing the sets of derivable formulas of the corresponding calculi, and we define the supporting translations.

The product calculi are contraction-free and cut-free. We prove that all logical rules are height-preserving invertible and that contraction and cut are admissible. To round off the theory, we sketch a proof of completeness.

2 Preliminaries

2.1 One-sided labelled sequent calculi

We presuppose a fixed set PV of propositional letters. Modal *formulas* are built from the elements of PV using propositional connectives and modalities. In the multi-modal case we use modalities $\Box_1, \ldots, \Box_n, \Diamond_1, \ldots, \Diamond_n$, in the unimodal case also $\Box, \Diamond$. As usual, $\Box_i$ and $\Diamond_i$ are dual to each other. We will present a calculus in the Schütte-Tait style, similar to the GS-calculi in [16].

$$\text{(ax)}\ \frac{}{\Gamma, x\colon P, x\colon \neg P} \qquad (\wedge)\ \frac{\Gamma, x\colon A \qquad \Gamma, x\colon B}{\Gamma, x\colon A \wedge B} \qquad (\vee)\ \frac{\Gamma, x\colon A, x\colon B}{\Gamma, x\colon A \vee B}$$

$$(\Box)\ \frac{\Gamma, x\overline{R}y, y\colon A}{\Gamma, x\colon \Box A}\ !(y) \qquad (\Diamond)\ \frac{\Gamma, x\overline{R}y, x\colon \Diamond A, y\colon A}{\Gamma, x\overline{R}y, x\colon \Diamond A}$$

Table 1
The system **GS3K**

Hence we work with formulas in *negation normal form.* This means that our formulas contain no implication and the only negated subformulas are negated propositional variables. Propositional variables P are also called *positive literals*, while the corresponding $\neg P$ is a *negative literal.* Negation *for non-atoms* is given by

$$\neg\Box_i A :\equiv \Diamond_i\neg A \quad \neg\Diamond_i A :\equiv \Box_i\neg A \quad \neg\Box A :\equiv \Diamond\neg A \quad \neg\Diamond A :\equiv \Box\neg A$$

$$\neg\neg P :\equiv P \quad \neg(A \wedge B) :\equiv \neg A \vee \neg B \quad \neg(A \vee B) :\equiv \neg A \wedge \neg B$$

We use $A \to B$ as a shorthand for $\neg A \vee B$. In two-sided sequent derivations of formulas in negation normal form, the left rules for logical symbols will never be used. In the one-sided style, these rules are removed completely, and negative literals are kept in the sole multiset corresponding to the right part of two-sided sequents. Thus the number of rules is reduced drastically, and what's more, the redundancy caused by pairs of dual rules (e.g. right $\wedge$ / left $\vee$) is removed.

Labelled formulas have the form $x : A$ where A is a modal formula and x is taken from a fixed set of labels. As weakening and contraction shall be absorbed into the logical rules, we start with the propositional, cut-free part of the calculus **GS3** in [16]. Prefixing formulas with labels, we obtain the non-modal part of the calculus **GS3K** in Table 1. Originally, Negri's labelled calculi are written in a two-sided style. Then sequents would have the form $\Gamma \Rightarrow \Delta$ where Γ, Δ are multisets of labelled formulas, and relational atoms have the form xRy. Logical axioms for relational atoms are not necessary. If they are not included, the atoms xRy occur on the left of the sequent arrow only. A two-sided sequent $A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n$ corresponds to the multiset $\neg A_1, \ldots, \neg A_m, B_1, \ldots, B_n$ in the one-sided style. With this transformation, the relational atoms would occur <u>negated</u> only in the one-sided system. We rather introduce relational symbols $\overline{R}$ or $\overline{R}_i$ for the complement relation right from the beginning and avoid negation. So relational atoms in the one-sided style have the form $x\overline{R}y$ or $x\overline{R}_iy$, *judgements* are relational atoms or labelled formulas, and a sequent is a multiset of judgements. Transferring Negri's modal rules to the one-sided style, we obtain the system **GS3K** in Table 1. Here $!(y)$ abbreviates the usual eigenvariable condition that y does not occur in the conclusion.

Adding relational rules, one can construct labelled calculi for a large variety of modal systems. The general method presented in [11] applies to normal modal logics which are characterised by universal axioms or, more generally, geometric implications as frame conditions. The two-sided versions of the resulting systems are studied in detail in [11]. As an illustration of the one-sided

$$(\text{ax})\ \frac{}{\Gamma, x\colon P, x\colon \neg P} \qquad (\wedge)\ \frac{\Gamma, x\colon A \qquad \Gamma, x\colon B}{\Gamma, x\colon A \wedge B} \qquad (\vee)\ \frac{\Gamma, x\colon A, x\colon B}{\Gamma, x\colon A \vee B}$$

$$(\Box)\ \frac{\Gamma, y\colon A}{\Gamma, x\colon \Box A}\ !(\mathrm{y}) \qquad (\Diamond)\ \frac{\Gamma, x\colon \Diamond A, y\colon A}{\Gamma, x\colon \Diamond A}$$

Table 2
The system **GS3S5u**

analogues, we present the rules obtained by direct transformation of the properties reflexivity, symmetry, and transitivity:

$$\frac{\Gamma, x\overline{R}x}{\Gamma}\ (\text{ref}) \qquad \frac{\Gamma, x\overline{R}z, x\overline{R}y, y\overline{R}z}{\Gamma, x\overline{R}y, y\overline{R}z}\ (\text{trans}) \qquad \frac{\Gamma, y\overline{R}x, x\overline{R}y}{\Gamma, x\overline{R}y}\ (\text{symm})$$

Note however that in some cases further rules have to be added to satisfy the *closure condition*. In the one-sided style, the geometric rule scheme has the form

$$\frac{\Gamma, U_1, \ldots, U_k, V_1^1, \ldots, V_{l_1}^1 \qquad \ldots \qquad \Gamma, U_1, \ldots, U_k, V_1^m, \ldots, V_{l_m}^m}{\Gamma, U_1, \ldots, U_k}\ !(\bar{x})$$

where the U_i and V_j^i are relational atoms and are called the *principal atoms* of that rule. The closure condition now refers to inferences obtained from instances of this scheme by contracting several occurrences of principal formulas of the conclusion and the corresponding occurrences in the premisses into one. It postulates that these should also be inferences of the system. The case that R is the universal relation could be formalised by a rule which allows to remove any atom $x\overline{R}y$. In that case, however, it would be more natural to remove relational atoms completely and modify the modal rules accordingly. The resulting system **GS3S5u** is shown in Table 2. These one-sided versions have already been used in [1] as the starting point for the development of a calculus. Here, they will be combined to product calculi in Sec. 3.

2.2 Products and relativised products of modal logics

The binary product combines two frames for uni- or multi-modal logics. More generally, one can consider the higher dimensional product of n frames. To simplify notation, we discuss the products of unimodal frames only. As a shorthand, we use $m..n$ for $\{m, \ldots, n\}$ where $m, n \in \mathbb{N}$. For $m > n$, $m..n$ denotes the empty set. Given $n \geq 2$, the *product* of frames $\mathcal{F}_i = (\mathcal{W}_i, \mathcal{R}_i)$, $i \in 1..n$, is the n-frame

$$\mathcal{F}_1 \times \ldots \times \mathcal{F}_n = (\mathcal{W}_1 \times \ldots \times \mathcal{W}_n, \hat{\mathcal{R}}_1, \ldots, \hat{\mathcal{R}}_n)$$

where, for each $i \in 1..n$, the binary relation $\hat{\mathcal{R}}_i$ on $\mathcal{W}_1 \times \ldots \times \mathcal{W}_n$ is given by

$$(u_1, \ldots, u_n)\hat{\mathcal{R}}_i(v_1, \ldots, v_n) \text{ iff } u_i \mathcal{R}_i v_i \text{ and } u_j = v_j, \text{ for } j \neq i.$$

Given n modal logics $\mathcal{L}_i$ formulated in languages that have no modal operator in common, the *product of* $\mathcal{L}_1, \ldots, \mathcal{L}_n$ is the modal logic determined by the class of product frames $\mathcal{F}_1 \times \ldots \times \mathcal{F}_n$ where the $\mathcal{F}_i$ are frames for $\mathcal{L}_i$. For example, $\mathbf{K}^n = \underbrace{\mathbf{K} \times \ldots \times \mathbf{K}}_{n}$ is the logic determined by all n-dimensional product frames, and $\mathbf{S5}^n$ is the logic determined by product frames $\mathcal{F}_1 \times \ldots \times \mathcal{F}_n$ where all $\mathcal{F}_i$ are equivalence frames. The class of products of universal frames $(W_i, W_i \times W_i)$ also determines the logic $\mathbf{S5}^n$. For the axiomatisations of products, the following *commutator axioms* are used:[2]

$$\begin{array}{ll} com^l{}_{i,j} :\equiv \Diamond_j \Diamond_i P \to \Diamond_i \Diamond_j P & com^r{}_{i,j} :\equiv \Diamond_i \Diamond_j P \to \Diamond_j \Diamond_i P (\equiv com^l{}_{j,i}) \\ com_{i,j} :\equiv com^l{}_{i,j} \wedge com^r{}_{i,j} & chr_{i,j} :\equiv \Diamond_i \Box_j P \to \Box_j \Diamond_i P \end{array}$$

The *commutator* $[\mathcal{L}_1, \ldots, \mathcal{L}_n]$ of modal logics $\mathcal{L}_i$, $i = 1, \ldots, n$, is the smallest n-modal logic containing all the $\mathcal{L}_i$ and the axioms $com_{i,j}$ and $chr_{i,j}$ for all $i, j \in 1..n$, $i \neq j$. It is easy to see that the axioms $com_{i,j}$ and $chr_{i,j}$ hold in all product frames. So the product of Kripke complete modal logics is an extension of the commutator. If the converse is also true, $\mathcal{L}_1, \ldots, \mathcal{L}_n$ is called *product-matching.* If $\mathcal{L}_1, \mathcal{L}_2$ are Kripke complete and Horn axiomatisable logics then $\mathcal{L}_1, \mathcal{L}_2$ is product-matching (see [8], Th. 21, and [2], Ch. 5 for a proof). For $n \geq 3$, any n-modal logic $\mathcal{L}$ such that $\mathbf{K}^n \subseteq \mathcal{L} \subseteq \mathbf{S5}^n$ is not finitely axiomatisable (see [8], Th. 25, and [6]). Now let $n \geq 2$, $\mathcal{K}$ be a class of subframes of n-ary product frames, and $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be Kripke complete unimodal logics formulated in languages that have no modal operator in common. The $\mathcal{K}$-*relativised product* $(\mathcal{L}_1 \times \ldots \times \mathcal{L}_n)^{\mathcal{K}}$ is the logic determined by the class of all subframes $\mathcal{G}$ of product frames $\mathcal{F}_1 \times \ldots \times \mathcal{F}_n$ where each $\mathcal{F}_i$ is a frame for $\mathcal{L}_i$, and $\mathcal{G} \in \mathcal{K}$. Note that the usual product is the special case where $\mathcal{K}$ contains all products of frames for $\mathcal{L}_i$. Let SF_n denote the class of all subframes of n-ary product frames. The SF_n-relativised products are also called *arbitrarily relativised products.* For $N \subseteq 1..n$, an n-ary N-*expanding relativised product frame* (see [10]) is a subframe $\mathcal{G} = (W, \ldots)$ of a product frame $\mathcal{F} = \mathcal{F}_1 \times \ldots \times \mathcal{F}_n$, of frames $\mathcal{F}_i = (W_i, R_i)$, $i \in 1..n$, where $(u_1, \ldots, u_{j-1}, v, u_{j+1}, \ldots, u_n) \in W$ for all $j \in N$, $(u_1, \ldots, u_n) \in W$, $v \in W_j$ satisfying $u_j R_j v$. Let EX_n^N be the class of all n-ary N-expanding relativised product frames. These determine the n-ary N-*expanding relativised product.* Axiomatisations for some expanding products can be found in [10]. An overview of results concerning products and other methods of combination can be found in [8].

3 Products of calculi

3.1 Language and rules

Components of the construction. We will combine n labelled calculi $C_1, \ldots, C_n$ and extend the result by further relational rules. We confine ourselves to unimodal logics and assume that the i-th calculus uses modalities $\Box_i, \Diamond_i$ where $\Box_1, \ldots, \Box_n, \Diamond_1, \ldots, \Diamond_n$ are pairwise distinct. We presuppose pairwise distinct relational symbols $R_1, \ldots, R_n$ and assume that the calculus C_i for logic $\mathcal{L}_i$ is formulated using symbol R_i. The case of **GS3S5u**, however,

[2] Here $\equiv$ stands for syntactic identity.

shall be included. To simplify notation, we assume that for some $r \in 0..n$ the first r calculi use the relation symbol R_i, whereas the remaining ones are copies of **GS3S5u**. The given calculi should follow the style in Sec. 2.1. In particular, the logical rules are the ones presented there, and the relational rules follow the given rule scheme and satisfy the closure condition. For the remaining section, we let $n \in \mathbb{N}_+$, $r \in 0..n$, and fix calculi $C_1, \ldots, C_n$ as just described. We use L_i for the corresponding infinite set of labels and add a further infinite set L of labels for the product calculus. For our convenience, we assume that the $L, L_1, \ldots, L_n$ are pairwise disjoint. This is not strictly necessary, as we can deduce the type of every occurrence of a label in a judgement, but it helps to shorten conditions like "x does not occur *as an L_i-label* in the sequent Γ" and similar. To deal with relativisations, the language will in general be extended by a unary predicate symbol $\overline{W}$ which is applied to labels in L. The symbol $\overline{W}$ stands for the complement of the relation "is-a-world" and is used to formalise conditions of the form "if α is a world then …". Properties of $\overline{W}, \overline{R}_1, \ldots \overline{R}_r$ can be described by further relational rules, see below.

The structure of labels. To deal with an n-dimensional product logic, we choose $L = L_1 \times \ldots \times L_n$. In order to obtain a compact and uniform notation for the calculus, we define the *access* and *update* operations by $(x_1, \ldots, x_n)(i) := x_i$ and $(x_1, \ldots, x_n)[i \leftarrow y] := (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$ for $(x_1, \ldots, x_n) \in L$, $i \in 1..n$ and $y \in L_i$. Remember that the elements of L_i are the labels of calculus C_i, typically primitive symbols. The labels $\alpha \in L$, however, are compound and can be considered as terms built from label "variables" $x_1, \ldots, x_n$ in $\bigcup_{i=1}^n L_i$. The notion *occurs in* and the operation *substitution* refer to this feature of L-labels. Apart from that, the system does not rely on the tuple notation but only on the fact that access and update satisfy the following conditions:

- $\alpha[i \leftarrow x](i) = x$ for all $\alpha \in L, i \in 1..n, x \in L_i$
- $\alpha[i \leftarrow x](j) = \alpha(j)$ for all $\alpha \in L, i, j \in 1..n, x \in L_i$ so that $i \neq j$
- $\forall i \in \{1, \ldots, n\}(\alpha(i) = \beta(i)) \rightarrow \alpha = \beta$ for all $\alpha, \beta \in L$

We use $x, x_1, x', y, z, \ldots$ for elements of $\bigcup_{i=1}^n L_i$ and $\alpha, \alpha_1, \alpha', \beta, \gamma, \ldots$ for elements of L. Let $x_1, \ldots, x_k$ be pairwise distinct labels in $\bigcup_{i=1}^n L_i$ and $i_1, \ldots, i_k \in 1..n$ so that $x_\nu \in L_{i_\nu}$ for all $\nu \in 1..k$. A finite set $\theta = \{y_1/x_1, \ldots, y_k/x_k\}$ where $y_\nu \in L_{i_\nu}$ for every $\nu \in 1..k$ is called *finite label substitution.* (We do not exclude $y_\nu \equiv x_\nu$.) For any expression E, $E\theta$ is the result of substituting simultaneously every occurrence of x_ν by y_ν, $\nu = 1, \ldots, k$. We abbreviate $\alpha[i \leftarrow x][j \leftarrow y]$ by $\alpha[i, j \leftarrow x, y]$ if i, j are distinct.

Judgements. *Labelled formulas* now have the form $\alpha : A$ where $\alpha \in L$ and A is a modal formula. *Judgements* are labelled formulas or relational atoms of the form $x\overline{R}_i y$ where $x, y \in L_i$ or $\overline{W}\alpha$ where $\alpha \in L$. The first kind of relational atoms is called R-relational. A *sequent* Γ is a finite multiset of judgements. The notion *occurs in* and the substitution operation are extended to judgements and sequents in the straightforward way. Due to our choice of label sets, the condition that the atom $\overline{W}\alpha$ or some labelled formula $\alpha : A$ occurs in some sequent Γ can be abbreviated to "α occurs in Γ" for $\alpha \in L$.

Definition 3.1 Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be frames, $\mathcal{F}_i = (\mathcal{W}_i, \mathcal{R}_i)$ for all $i \in 1..n$, and $\mathcal{F} = (\mathcal{W}, \hat{\mathcal{R}}_1, \ldots, \hat{\mathcal{R}}_n)$ be a subframe of their product. Furthermore let $\mathcal{V} : \mathcal{W} \to \mathrm{PV} \to \mathbb{B}$ be a valuation, and $\mathcal{M} = (\mathcal{F}, \mathcal{V})$. For every $w \in \mathcal{W}$, we write $w \models_{\mathcal{M}} A$ for "A is true at the world w of $\mathcal{M}$". Let $\ell_i : L_i \to \mathcal{W}_i$, $i = 1, \ldots, n$, be mappings, set $\bar{\ell} = \ell_1, \ldots, \ell_n$, and define $\bar{\ell}(\alpha) = (\ell_1(\alpha(1)), \ldots, \ell_n(\alpha(n)))$ for every $\alpha \in L$. Then, based on the validity of formulas, we define the validity of judgements and sequents as follows:

$$\begin{array}{lcl}
(\mathcal{M}, \bar{\ell}) \models \overline{W}\alpha & \iff & \bar{\ell}(\alpha) \notin \mathcal{W} \\
(\mathcal{M}, \bar{\ell}) \models \alpha : A & \iff & \text{if } \bar{\ell}(\alpha) \in \mathcal{W} \text{ then } \bar{\ell}(\alpha) \models_{\mathcal{M}} A \\
(\mathcal{M}, \bar{\ell}) \models x\overline{R}_i y & \iff & (\ell_i(x), \ell_i(y)) \notin \mathcal{R}_i \text{ for } i \in 1..n \\
(\mathcal{M}, \bar{\ell}) \models \Gamma & \iff & (\mathcal{M}, \bar{\ell}) \models J \text{ for some judgement } J \text{ in } \Gamma
\end{array}$$

As we use it to formalise conditions on worlds, it is most natural to introduce a corresponding predicate and write $\overline{W}\alpha$ but $\alpha : \bot$ where $\bot$ is any contradictory proposition has the same interpretation.

Relational rules. Let T_i stand for the relational rules of calculus C_i, $i \in 1..r$. These shall be included, where now the context Γ consists of judgements in the extended language. Furthermore, we want to add rules that refer to $\overline{W}$ and possibly several $\overline{R}_i$. It is required that the additional rules have the form

$$\frac{\Gamma, U_1, \ldots, U_k, V_1^1, \ldots, V_{l_1}^1 \qquad \ldots \qquad \Gamma, U_1, \ldots, U_k, V_1^m, \ldots, V_{l_m}^m}{\Gamma, U_1, \ldots, U_k}\ !(\bar{x})?(\bar{\alpha})$$

where $\bar{x}, \bar{\alpha}$ are lists of labels, the U_i are R-relational and V_j^i arbitrary relational atoms. These atoms are called *principal*. For the general results below, we need no further restriction on the V_ν^μ. The intended application, however, is describing relativised products. Typically, the condition on subframes consists in postulating that some tuples are worlds if certain preconditions are fulfilled. Then the V_ν^μ would simply be of the form $\overline{W}\beta$. Furthermore, none of the systems discussed in 3.2 makes use of eigenvariables. It is understood that every instance of the rule scheme that satisfies the side conditions is an accepted inference. The $!(\bar{x})$ abbreviates, as usual, the eigenvariable condition for the label $\bar{x} = x_1, \ldots, x_k$, while $?(\bar{\alpha})$ stands for the additional requirement that the elements of $\bar{\alpha} = \alpha_1, \ldots, \alpha_\ell$ do occur in the conclusion. The closure condition now refers to inferences obtained from instances of this rule scheme, in which several $U_{j_1}, \ldots, U_{j_o}$ become identical by substitution. We postulate that the result of contracting these into one, and also contracting the corresponding occurrences in the premisses into one, is an inference of the system. Then the calculi described below depend on

- $n \in \mathbb{N}_+$, the dimension of the product,
- some $r \in \mathbb{N}$ so that $0 \leq r \leq n$ which determines the number of calculi that are no copies of **GS3S5u**,

$$(\mathrm{ax})\ \frac{}{\Gamma, \alpha\colon P, \alpha\colon \neg P} \qquad (\vee)\ \frac{\Gamma, \alpha\colon A, \alpha\colon B}{\Gamma, \alpha\colon A \vee B} \qquad (\wedge)\ \frac{\Gamma, \alpha\colon A \qquad \Gamma, \alpha\colon B}{\Gamma, \alpha\colon A \wedge B}$$

For all $i \in 1..r$:

$$(\Box_i)\ \frac{\Gamma, \overline{W}\alpha, \alpha(i)\overline{R}_i x, \alpha[i \leftarrow x]\colon A}{\Gamma, \alpha\colon \Box_i A}\ !(x)$$

For all $i \in (r+1)..n$:

$$(\Box_i)\ \frac{\Gamma, \overline{W}\alpha, \alpha[i \leftarrow x]\colon A}{\Gamma, \alpha\colon \Box_i A}\ !(x)$$

For all $i \in 1..r$:

$$(\Diamond_i)\ \frac{\Gamma, \alpha(i)\overline{R}_i x, \alpha\colon \Diamond_i A, \alpha[i \leftarrow x]\colon A}{\Gamma, \alpha(i)\overline{R}_i x, \alpha\colon \Diamond_i A}\ ?(\alpha[i \leftarrow x])$$

For all $i \in (r+1)..n$:

$$(\Diamond_i)\ \frac{\Gamma, \alpha\colon \Diamond_i A, \alpha[i \leftarrow x]\colon A}{\Gamma, \alpha\colon \Diamond_i A}\ ?(\alpha[i \leftarrow x])$$

Table 3
The logical rules of $\mathbf{LS}^{n,r}_{W\times}(T)$

- the decision whether to include $\overline{W}$ or not, and
- some set $T \supseteq \bigcup_{i=1}^{r} T_i$ of relational rules as described above.

The resulting **l**abelled **s**ystem will be denoted by $\mathbf{LS}^{n,r}_{W\times}(T)$ or $\mathbf{LS}^{n,r}_{\times}(T)$ respectively. Slightly misusing notation, as the choice of T depends on the language, we write $\mathbf{LS}^{n,r}_{(W)\times}(T)$ to talk about both variants.

Logical rules. The logical rules of $\mathbf{LS}^{n,r}_{W\times}(T)$ are given in Table 3. The relational part T may introduce further axioms, in which case we refer to the axiom scheme presented in Tab. 3 as *logical axioms.* The side condition in $(\Diamond_i)$ is necessary for soundness, in contrast to the extra $\overline{W}\alpha$ in the premiss of rule $(\Box_i)$. If desired, one could safely add also the variant of the $\Box_i$-rule without $\overline{W}\alpha$. The version above, however, enables us to contract some $\overline{W}\alpha$ into the newly built $\alpha : \Box_i A$, and we will use this possibility below. Note that every relational atom $x\overline{R}_i y$ and every label $\alpha \in L$ in the conclusion of a logical rule occurs also in all premisses.

The world predicate $\overline{W}$ combined with the side conditions "?" are introduced for relativisations. For the non-relativised products, we present the simplified variants $\mathbf{LS}^{n,r}_{\times}(T)$ obtained by removing them. They are given as follows: The rules (ax), $(\wedge)$ and $(\vee)$ are the same as in $\mathbf{LS}^{n,r}_{W\times}(T)$, except that now the context does not contain atoms $\overline{W}\alpha$. The rules for the modalities are given in Tab. 4. We write $\mathbf{LS}^{n,r}_{(W)\times}(T) \vdash \Gamma$ if Γ is derivable in $\mathbf{LS}^{n,r}_{W\times}(T)$, furthermore we use $\vdash^k$ if there is a derivation of height $\leq k$. A modal *formula* A is derivable if the sequent $\alpha : A$ is derivable for some $\alpha \in L$.

Lemma 3.2 (Soundness) *If* $\mathbf{LS}^{n,r}_{W\times}(T) \vdash \Gamma$ *then* $(\mathcal{M}, \bar{\ell}) \models \Gamma$ *for all models* $\mathcal{M} = (\mathcal{F}, \mathcal{V})$ *based on subframes* $\mathcal{F}$ *of product frames that satisfy all frame conditions determined by* T *and all appropriate assignment functions* $\bar{\ell}$.

Proof By induction on the height of the derivation. Let $\mathcal{F}_i = (\mathcal{W}_i, \mathcal{R}_i)$ for all

For all $i \in 1..r$:

$$(\Box_i)\ \frac{\Gamma, \alpha(i)\overline{R}_i x, \alpha[i \leftarrow x] \colon A}{\Gamma, \alpha \colon \Box_i A}\ !(x)$$

$$(\Diamond_i)\ \frac{\Gamma, \alpha(i)\overline{R}_i x, \alpha \colon \Diamond_i A, \alpha[i \leftarrow x] \colon A}{\Gamma, \alpha(i)\overline{R}_i x, \alpha \colon \Diamond_i A}$$

For all $i \in (r+1)..n$:

$$(\Box_i)\ \frac{\Gamma, \alpha[i \leftarrow x] \colon A}{\Gamma, \alpha \colon \Box_i A}\ !(x)$$

$$(\Diamond_i)\ \frac{\Gamma, \alpha \colon \Diamond_i A, \alpha[i \leftarrow x] \colon A}{\Gamma, \alpha \colon \Diamond_i A}$$

Table 4
The rules for the modalities in $\mathbf{LS}^{n,r}_{\times}(T)$

$i \in 1..n$, $\mathcal{F} = (\mathcal{W}, \ldots)$ and $\bar{\ell} = \ell_1, \ldots, \ell_n$. We consider the case $\Diamond_i$, $i \in 1..r$. Let $\Gamma = \Gamma', \alpha(i)\overline{R}_i x, \alpha : \Diamond_i A$. By IH $(\mathcal{M}, \bar{\ell}) \models \Gamma, \alpha[i \leftarrow x] : A$. Obviously, Γ holds if $(\ell_i(\alpha(i)), \ell_i(x)) \notin \mathcal{R}_i$. As $\alpha[i \leftarrow x]$ occurs in Γ, that sequent holds if $\bar{\ell}(\alpha[i \leftarrow x]) \notin \mathcal{W}$. Otherwise the validity of $\alpha[i \leftarrow x] : A$ implies validity of $\alpha : \Diamond_i A$, hence again validity of Γ. □

3.2 Instantiations of the scheme

In this section we consider instantiations of the general scheme to deal with products and several kinds of relativised products. Let T_i again stand for the relational rules of calculus C_i and $T' := \bigcup_{i=1}^{r} T_i$.

Arbitrary and expanding relativisations. To deal with relativisations, we need the version $\mathbf{LS}^{n,r}_{W\times}(T)$ where the language includes the predicate symbol $\overline{W}$ and the rules refer to it. For arbitrarily relativised products, we add no rule for $\overline{W}$ in the calculus. For the expanding relativisations, we introduce the following relational axioms:

$$(i\mathrm{EX})\ \frac{\Gamma, \alpha(i)\overline{R}_i y, \overline{W}\alpha[i \leftarrow y]}{\Gamma, \alpha(i)\overline{R}_i y}\ ?(\alpha) \qquad\qquad (i\mathrm{EXu})\ \frac{\Gamma, \overline{W}\alpha[i \leftarrow y]}{\Gamma}\ ?(\alpha)$$

For $N \subseteq 1..n$, let $T^{N\text{-EX}}$ be obtained from T' by adding the rules (iEX) for $i \in N \cap 1..r$ and (iEXu) for $i \in N \cap (r+1)..n$. To illustrate the application of these frame rules we present derivations of the axioms of the expanding commutator.

Lemma 3.3 *Let $N \subseteq 1..n$. The formulas $com^l_{i,j}$ and $chr_{i,j}$ are derivable in $\boldsymbol{LS}^{n,r}_{W\times}(T^{N-EX})$ if $i \in N$.*

Proof Let $i \in N$. Let P be a propositional symbol, $\alpha \in L$, and $L_i \ni x \neq \alpha(i)$, $L_j \ni y \neq \alpha(j)$ for some $j \in 1..n$, $i \neq j$. We present the derivations for the case that $i, j \leq r$. As abbreviation, let $\Gamma := \overline{W}\alpha, \alpha(j)\overline{R}_j y, \overline{W}\alpha[j \leftarrow y], \alpha(i)\overline{R}_i x$. Note that due to ($i$EX) we can infer Γ, Δ from $\Gamma, \Delta, \overline{W}\alpha[i \leftarrow x]$. So we get the following derivation in $\mathbf{LS}^{n,r}_{W\times}(T^{N\text{-EX}})$:

$$
(\vee)\;\dfrac{(\Box_j)\;\dfrac{(\Box_i)\;\dfrac{(i\mathrm{EX})\;\dfrac{(\Diamond_i)\;\dfrac{(\Diamond_j)\;\dfrac{\Gamma, \overline{W}\alpha[i\!\leftarrow\! x], \alpha[i,j\!\leftarrow\! x,y] : \neg P, \alpha : \Diamond_i\Diamond_j P, \alpha[i\!\leftarrow\! x] : \Diamond_j P, \alpha[i,j\!\leftarrow\! x,y] : P}{\Gamma, \overline{W}\alpha[i\!\leftarrow\! x], \alpha[i,j\!\leftarrow\! x,y] : \neg P, \alpha : \Diamond_i\Diamond_j P, \alpha[i\!\leftarrow\! x] : \Diamond_j P}}{\Gamma, \overline{W}\alpha[i\!\leftarrow\! x], \alpha : \Diamond_i\Diamond_j P, \alpha[i,j\!\leftarrow\! x,y] : \neg P}}{\Gamma, \alpha : \Diamond_i\Diamond_j P, \alpha[i,j\!\leftarrow\! x,y] : \neg P}}{\overline{W}\alpha, \alpha(j)\overline{R}_j y, \alpha : \Diamond_i\Diamond_j P, \alpha[j\!\leftarrow\! y] : \Box_i\neg P}}{\alpha : \Box_j\Box_i\neg P, \alpha : \Diamond_i\Diamond_j P}}{\alpha : \Box_j\Box_i\neg P \vee \Diamond_i\Diamond_j P}
$$

The derivation of $\mathrm{chr}_{i,j}$ can be found in the appendix. The remaining cases are similar. □

Note that $\mathrm{com}^l_{j,i}$ is also derivable if $i > r$ but not in general for $i \leq r$.

Non-relativised products. As the product is the special case of the relativised product where every tuple is a world, one can use $\mathbf{LS}^{n,r}_{W\times}(T^{\mathrm{all}})$ where T^{all} is obtained from T' by adding the rule:

$$
(\text{all-}W)\;\dfrac{\Gamma, \overline{W}\alpha}{\Gamma}
$$

This allows for a unified treatment of relativised and non-relativised products. In the latter case however, the symbol $\overline{W}$ is dispensable. The modal rules for the simplified version without $\overline{W}$ were given in Table 4. In Sec. 4 we show that the two systems for non-relativised products are equivalent. In $\mathbf{LS}^{n,r}_{\times}(T')$, all axioms $\mathrm{com}^l_{i,j}$, $\mathrm{com}^r_{i,j}$ and $\mathrm{chr}_{i,j}$, $i \neq j$, are derivable. To see this, reinspect the derivations in 3.3 and simplify them. Moreover, it is straightforward how to transform derivations in C_i in derivations in $\mathbf{LS}^{n,r}_{\times}(T')$ that prove the same modal formula. For conservativity of the extension, we remark that the derivations in $\mathbf{LS}^{n,r}_{\times}(T')$ have the following properties:

Lemma 3.4 *Let Γ be a sequent and $I \subseteq 1..n$ so that $i \in I$ for all modalities $\Box_i, \Diamond_i$ in Γ. Consider a derivation d of Γ in $\boldsymbol{LS}^{n,r}_{(W)\times}(T')$. Then for every labelled formula $\beta : B$ in d, B is a subformula of some formula in Γ and for all $j \in 1..n$, $j \notin I$, there is some labelled formula $\alpha : A$ in Γ for which $\beta(j) = \alpha(j)$.*

Proof Inspection of the rules of the calculus. □

Due to this fact, every derivation in $\mathbf{LS}^{n,r}_{\times}(T')$ of some $\alpha : A$ where A contains only $\Diamond_i, \Box_i$ can be transformed in a straightforward way into a derivation in C_i with endsequent $\alpha(i) : A$.

3.3 Basic proof-theoretic properties

For the remainder of this section, we assume a fixed set $T \supseteq \bigcup_{i=1}^r T_i$ of rules to be given which satisfies the conditions of Sec. 3.1.

Lemma 3.5 *The following holds for $\boldsymbol{LS}^{n,r}_{(W)\times}(T)$:*

(i) (renaming) *Let d be a derivation with endsequent Γ, $i \in 1..n$, and $x, y \in L_i$ so that y does not occur in d. Then replacing every occurrence of x in d by y yields a derivation of the endsequent $\Gamma\{y/x\}$.*

(ii) (label substitution) *Let θ be a finite label substitution. Then* $\vdash^k \Gamma$ *implies* $\vdash^k \Gamma\theta$.

(iii) (weakening) $\vdash^k \Gamma \Longrightarrow \vdash^k \Gamma, J$ *for every judgement* J

Proof Straightforward induction on the height of the given derivation. We use Fact (i) in the proofs of (ii),(iii) to avoid a clash with eigenvariables. □

Due to (renaming), we can always assume that the eigenvariables in a given derivation d are only used as eigenvariable for one particular inference and do not occur in any further judgement, sequent or given derivation other than d. We make use of this fact in many of the proofs below, without mentioning it explicitly.

Lemma 3.6 (Relational contraction) *Let Γ be a sequent, $x, y \in L_i$, and $\alpha \in L$.*

(i) (R-contraction) $\vdash^k \Gamma, x\overline{R}_i y, x\overline{R}_i y \Longrightarrow \vdash^k \Gamma, x\overline{R}_i y$

(ii) (W-contraction) $\vdash^k \Gamma, \overline{W}\alpha \Longrightarrow \vdash^k \Gamma$ *if α occurs in Γ.*

Proof By induction on k, using the closure condition for relational rules. □

Theorem 3.7 (Invertibility of logical rules) *The following holds for the calculus $\boldsymbol{LS}^{n,r}_{(W)\times}(T)$:*

(i) ($\wedge$-inversion) $\vdash^k \Gamma, \alpha\colon A \wedge B$ *implies* $\vdash^k \Gamma, \alpha\colon A$ *and* $\vdash^k \Gamma, \alpha\colon B$

(ii) ($\vee$-inversion) $\vdash^k \Gamma, \alpha\colon A \vee B$ *implies* $\vdash^k \Gamma, \alpha\colon A, \alpha\colon B$

(iii) ($\Box_i$-inversion) $\vdash^k \Gamma, \alpha\colon \Box_i A$ *implies* $\vdash^k \Gamma, \alpha(i)\overline{R}_i x, (\overline{W}\alpha,)\alpha[i \leftarrow x]\colon A$ *for every $x \in L_i$ if $i \in 1..r$, and* $\vdash^k \Gamma, (\overline{W}\alpha,)\alpha[i \leftarrow x] : A$ *if $i \in (r+1)..n$.*

Proof By induction on the height of the derivation. Note that in the W-version the sequent(s) after inversion contain every $\beta \in L$ that is in the sequent before inversion. So context conditions $?(\beta)$ are not destroyed. □

Inversion is helpful in proofs of non-derivability. For example, we can show the non-derivability of the formulas $\mathrm{com}^{\ell}_{i,j}$, $i \neq j$, $i, j \in 1..r$, in $\mathbf{LS}^{n,r}_{W\times}(\bigcup_{i=1}^{r} T_i)$ as follows: Suppose we had a derivation of $\alpha : \Box_j\Box_i\neg P \vee \Diamond_i\Diamond_j P$. Then by (inversion) and (weakening) we obtain a derivation of

$$\overline{W}\alpha, \overline{W}\alpha[j \leftarrow y], \alpha(j)\overline{R}_j y, \alpha(i)\overline{R}_i x,$$
$$\alpha[i, j \leftarrow x, y] : \neg P, \alpha : \Diamond_i\Diamond_j P, \alpha : \Diamond_j P, \alpha : P, \alpha[j \leftarrow y] : P$$

for fresh $x \in L_i, y \in L_j$. As the rules in $\bigcup_{i=1}^{r} T_i$, read from bottom to top, add only R-relational atoms, every sequent in that derivation satisfies the side condition $?(\beta)$ for the same expressions β. Hence every $(\Diamond_j)$- or $(\Diamond_i)$-rule, also read from bottom to top, adds no new labelled formulas. As a consequence, the topmost sequent contains the same labelled formulas as the endsequent, and hence it is no axiom.

Moreover, inversion is crucial to the proof of the admissibility of contraction.

Theorem 3.8 (Admissibility of contraction) *Let Γ be a sequent, A a formula and $\alpha \in L$. If $\boldsymbol{LS}^{n,r}_{(W)\times}(T) \vdash^k \Gamma, \alpha\colon A, \alpha\colon A$ then $\boldsymbol{LS}^{n,r}_{(W)\times}(T) \vdash^k \Gamma, \alpha\colon A$.*

Proof By induction on the height of the derivation. In the case that one of the distinguished occurrences of $\alpha : A$ is constructed, we use inversion combined with the IH. Furthermore, in the case of $(\Box_i)$, (R-contraction) (and (W-contraction)) is used. $\Box$

4 Comparison of the variations

Again, we fix $n \in \mathbb{N}_+$, $r \in 0..n$, and calculi $C_1, \dots, C_n$ as described in Sec. 2.1, let T_i, $i \in 1..r$, stand for the relational rules in C_i and $T' := \bigcup_{i=1}^{n} T_i$. Furthermore, let T^{all} be obtained from T' by adding the rule (all-W). This rule formalises the fact that every tuple is a world, and can be used to deal with the product as a special case of relativised products. For the non-relativised case however, we also introduced the simplification $\mathbf{LS}_{\times}^{n,r}(T')$ without $\overline{W}$-predicate.

Lemma 4.1 *Let Γ be a sequent not containing $\overline{W}$. Then Γ is derivable in $\boldsymbol{LS}_{W\times}^{n,r}(T^{all})$ if and only if it is derivable in $\boldsymbol{LS}_{\times}^{n,r}(T')$.*

Proof Consider first a $\mathbf{LS}_{W\times}^{n,r}(T^{\mathrm{all}})$-derivation. Removing all instances of the rule (all-W) and all occurrences of some $\overline{W}\beta$ yields a derivation in $\mathbf{LS}_{\times}^{n,r}(T')$. Now we turn to the converse direction and proceed by induction on the height of the given derivation. We present the constructions for the cases of the modalities for $i \in 1..r$.

$(\Box_i)$: Suppose $\Gamma', \alpha(i)\overline{R}_i x, \alpha[i \leftarrow x] : A$ is the endsequent of the derivation obtained by applying the IH. Using (weakening), we can add $\overline{W}\alpha$ to this, and subsequently apply $(\Box_i)$ to deduce $\Gamma', \alpha : \Box_i A$.

$(\Diamond_i)$: Suppose $\Gamma', \alpha : \Diamond_i A, \alpha(i)\overline{R}_i x, \alpha[i \leftarrow x] : A$ is the endsequent of the derivation obtained by applying the IH. Using (weakening), we add $\overline{W}\alpha[i \leftarrow x]$ to this, and, as $\alpha[i \leftarrow x]$ occurs in the result, apply $(\Diamond_i)$ to deduce the sequent $\Gamma', \alpha : \Diamond_i A, \alpha(i)\overline{R}_i x, \overline{W}\alpha[i \leftarrow x]$. Finally, $\overline{W}\alpha[i \leftarrow x]$ is removed by (all-W). $\Box$

As a consequence, the sets of derivable formulas in $\mathbf{LS}_{W\times}^{n,r}(T^{\mathrm{all}})$ and $\mathbf{LS}_{\times}^{n,r}(T')$ coincide.

The second type of variation discussed here concerns the formalisation of the S5 components. As **GS3S5u** offers a simplified version without relational symbol for the logic S5, we admitted that system as component calculus. In the definition of the product however, it is stressed that in general one has to consider *all* frames for the logics. So alternatively, we use systems based on the rules for equivalence. For $i \in (r+1)..n$, let T_i consist of the rules (ref), (symm), (trans) formulated for symbol $\overline{R}_i$. Furthermore let $T'_s := T' \cup \bigcup_{i=r+1}^{s} T_i$ for $s \in r..n$. Then the last $n - r$ components of $\mathbf{LS}_{(W)\times}^{n,n}(T'_n)$ are also systems for S5, the logical rules for all components follow the same pattern, and we do not have to distinguish the cases $i \leq r$ and $i > r$ when we *argue about* the product calculus. On the other hand, the components based on **GS3S5u**, which are used in $\mathbf{LS}_{(W)\times}^{n,r}(T')$, are obtained from $\mathbf{LS}_{(W)\times}^{n,n}(T'_n)$ by removing the relational symbols $\overline{R}_i$ for $i \in (r + 1)..n$ and the corresponding rules. They are simpler and hence preferable when we *work with* the calculus. For the comparison of these systems, we first prove the following auxiliary lemma.

Lemma 4.2 *Let $s \in (r+1)..n$ and Γ be a sequent so that $\boldsymbol{LS}^{n,s-1}_{\times}(T'_{s-1}) \vdash^k \Gamma$. Then there is a $\boldsymbol{LS}^{n,s-1}_{\times}(T'_{s-1})$-derivation with endsequent Γ and height $\leq k$ so that, for every instance 5*

$$(\star\star) \qquad \frac{\Gamma', \alpha : \Diamond_s A, \alpha[s \leftarrow x] : A}{\Gamma', \alpha : \Diamond_s A}$$

of $(\Diamond_s)$, the label x does occur in the conclusion.

Proof By induction on k. Consider the case in which the last inference has the form $(\star\star)$ where x does *not* occur in the conclusion. By (substitution) we obtain a derivation of height $\leq k-1$ where the endsequent is $\Gamma', \alpha : \Diamond_s A, \alpha : A$. By IH we can assume that the $\Diamond_s$-inferences in that derivation are as requested. Now we apply $(\Diamond_s)$ to build the derivation of Γ. The remaining cases are trivial. $\Box$

Definition 4.3 Let $s \in 1..n$, $L', L'' \subseteq L_s$, and Γ be a sequent. Then define

$$\begin{array}{lll} L_s(\Gamma) & = & \{x \in L_s \mid x \text{ occurs in } \Gamma\} \\ L'\overline{R}_s L'' & = & \text{the least multiset that contains all relational} \\ & & \text{atoms } x'\overline{R}_s x'' \text{ where } x' \in L' \text{ and } x'' \in L'' \end{array}$$

Lemma 4.4 *Let $s \in (r+1)..n$ and Γ be a sequent. Then $\boldsymbol{LS}^{n,s-1}_{(W)\times}(T'_{s-1}) \vdash \Gamma$ implies $\boldsymbol{LS}^{n,s}_{(W)\times}(T'_s) \vdash \Gamma, L_s(\Gamma)\overline{R}_s L_s(\Gamma)$.*

Proof First, we consider the calculus $\mathbf{LS}^{n,s-1}_{\times}(T'_{s-1})$. By Lem. 4.2 we can assume that, for every application $(\star\star)$ of a $(\Diamond_s)$-rule in the given derivation, the variable x does occur in the conclusion. Now we proceed by induction on the height of such a derivation. Let Γ be the endsequent and $\Pi = L_s(\Gamma)\overline{R}_s L_s(\Gamma)$. We distinguish cases according to the rule applied last. We present the cases of the modalities $(\Diamond_s)$ and $(\Box_s)$.

$(\Diamond_s)$: Assume that the last inference has the form $(\star\star)$. As x occurs in the conclusion, we have $L_s(\Gamma) = L_s(\Gamma', \alpha : \Diamond_s A, \alpha[s \leftarrow x] : A)$. By IH the sequent $\Gamma', \alpha : \Diamond_s A, \alpha[s \leftarrow x] : A, \Pi$ is derivable in $\mathbf{LS}^{n,s}_{\times}(T'_s)$. As $\alpha(s)\overline{R}_s x$ occurs in Π, we can deduce $\Gamma', \alpha : \Diamond_s A, \Pi$.

$(\Box_s)$: Assume the last inference is:

$$(\Box_s)\ \frac{\Gamma', \alpha[s \leftarrow x] : A}{\Gamma', \alpha : \Box_s A}\ !(x)$$

Let $\Pi' = L_s(\Gamma')\overline{R}_s L_s(\Gamma')$. By IH the sequent

$$\Gamma', \alpha[s \leftarrow x] : A, \Pi', \{x\}\overline{R}_s L_s(\Gamma'), L_s(\Gamma')\overline{R}_s\{x\}, x\overline{R}_s x$$

is derivable in $\mathbf{LS}^{n,s}_{\times}(T'_s)$. Now we apply (ref) to remove $x\overline{R}_s x$ and (symm) to remove all elements of $\{x\}\overline{R}_s L_s(\Gamma')$.

Subcase 1: $\alpha(s) \in L_s(\Gamma')$. Then, for every $y \in L_s(\Gamma') \setminus \{\alpha(s)\}$, the atom $y\overline{R}_s\alpha(s)$ is in Π', and we can remove $y\overline{R}_s x$ using (trans). Now we have a deduction of $\Gamma', \alpha[s\leftarrow x] : A, \Pi', \alpha(s)\overline{R}_s x$, and the only occurrences of x in this sequent are those explicitly mentioned. So we can apply $(\Box_s)$ to deduce Γ, Π'.

Subcase 2: $\alpha(s) \notin L_s(\Gamma')$. Then

$$\Pi = \Pi', L_s(\Gamma')\overline{R}_s\{\alpha(s)\}, \{\alpha(s)\}\overline{R}_s L_s(\Gamma'), \alpha(s)\overline{R}_s\alpha(s)$$

and we use (weakening) to obtain a derivation of:

$$\Gamma', \alpha[s\leftarrow x] : A, \Pi, \alpha(s)\overline{R}_s x, L_s(\Gamma')\overline{R}_s\{x\}$$

As, for every $y \in L_s(\Gamma')$, the sequent Π contains $y\overline{R}_s\alpha(s)$, we can apply (trans) to remove $y\overline{R}_s x$. Subsequently, we use $(\Box_s)$ to derive $\Gamma', \alpha : \Box_s A, \Pi$.

The case $\mathbf{LS}^{n,s-1}_{W\times}(T'_{s-1})$ is even simpler. The side condition for $(\Diamond_s)$ ensures that the substituted label x occurs in the conclusion. Moreover, for every instance of a $(\Box_s)$ with conclusion $\Gamma', \alpha : \Box_s A$, we know that α occurs in the premiss, hence we proceed as in the first subcase above. □

Theorem 4.5 *Let $s \in (r+1)..n$, A be a modal formula and $\alpha \in L$. Then* $\boldsymbol{LS}^{n,s-1}_{(W)\times}(T'_{s-1}) \vdash \alpha : A \Longleftrightarrow \boldsymbol{LS}^{n,s}_{(W)\times}(T'_s) \vdash \alpha : A$.

Proof Removing all relational atoms $x\overline{R}_s y$ and all instances of rules in $T'_s \setminus T'_{s-1}$ in a $\mathbf{LS}^{n,s}_{(W)\times}(T'_s)$-derivation, we obtain a derivation in $\mathbf{LS}^{n,s-1}_{(W)\times}(T'_{s-1})$. For the converse, we consider a derivation of $\alpha : A$ in $\mathbf{LS}^{n,s-1}_{(W)\times}(T'_{s-1})$ and apply Lem. 4.4 to obtain a derivation of $\alpha(s)\overline{R}_s\alpha(s), \alpha : A$ in $\mathbf{LS}^{n,s}_{(W)\times}(T'_s)$. Using (ref), we can deduce $\alpha : A$. □

Corollary 4.6 *Let A be a formula and $\alpha \in L$. Then* $\boldsymbol{LS}^{n,r}_{(W)\times}(T') \vdash \alpha : A \Longleftrightarrow \boldsymbol{LS}^{n,n}_{(W)\times}(T'_n) \vdash \alpha : A$.

5 Admissibility of cut

Next we turn to the cut rule. As we have shown in Sec. 4 that the variants for non-relativised products as well as the systems where $r < n$ are equivalent to some $\mathbf{LS}^{n,n}_{W\times}(\hat{T})$, we consider only the latter type of calculus. Note that the sound version of cut is

$$(\text{cut})\ \frac{\Gamma, \alpha : A \qquad \Pi, \alpha : \neg A}{\Gamma, \Pi}\ ?(\alpha)$$

with the side condition $?(\alpha)$. If both $\alpha : A$ and $\alpha : \neg A$ are true in a model $\mathcal{M}$ w.r.t. the labelling $\bar{\ell}$ then $\bar{\ell}(\alpha)$ is no world in $\mathcal{M}$. If α occurs in Γ, Π then this implies that Γ, Π holds.

Theorem 5.1 (Admissibility of cut) *If* $\boldsymbol{LS}^{n,n}_{W\times}(T) \vdash^k \Gamma, \alpha : A$ *and also* $\boldsymbol{LS}^{n,n}_{W\times}(T) \vdash^m \Pi, \alpha : \neg A$ *where α occurs in Γ, Π, then* $\boldsymbol{LS}^{n,n}_{W\times}(T) \vdash \Gamma, \Pi$

Proof By induction on A, side induction on $k + m$.

Case 1: $\alpha : A$ is not constructed in the last inference of the first derivation or $\alpha : \neg A$ is not constructed in the last inference of the second derivation. If that derivation consists of an axiom only, then Γ or Π is an axiom, hence Γ, Π

is. Otherwise, we apply the side induction hypothesis, and deduce Γ, Π in one step. That this is possible depends on the following facts:

- The side IH is applicable: If $\alpha : A$ is not principal in the last inference and α occurs in Γ, then α occurs also in the part Γ' of the premiss $\Gamma', \alpha : A$ of that last inference, similar for the second derivation.
- The potential context conditions of the last inference are not destroyed: Renaming eigenvariables first takes care of the eigenvariable condition. If we consider the first derivation and α does not occur in the part Γ' of the premiss $\Gamma', \alpha : A$, then it occurs in Π, as we postulated that it occurs in Γ, Π. A similar argument applies to the second derivation.

Case 2: Both $\alpha : A$ and $\alpha : \neg A$ are principal in the last inference. If A is a positive or negative literal then $\alpha : \neg A$ must be a labelled formula in Γ and $\alpha : A$ must be a labelled formula in Π. Hence Γ, Π is an axiom in this case. If the principal symbols in A and $\neg A$ are $\vee, \wedge$, we just have to apply the IH and (contraction). We present the more involved case that their principal symbols are modalities. W.l.o.g. the principal symbol in A is some $\Box_i$. Then the principal symbol in $\neg A$ is $\Diamond_i$. Let B be a formula and $x, y \in L_i$ so that the premiss of the first derivation is $\Gamma, \overline{W}\alpha, \alpha(i)\overline{R}_i x, \alpha[i \leftarrow x] : B$ and the premiss of the second is $\Pi, \alpha : \Diamond_i B, \alpha[i \leftarrow y] : \neg B$. (Note that $\alpha(i)\overline{R}_i y$ is an element of Π). Using (substitution) we obtain a derivation of $\Gamma, \overline{W}\alpha, \alpha(i)\overline{R}_i y, \alpha[i \leftarrow y] : B$. Combining the first derivation with the immediate subderivation of the second and applying SIH, we get a derivation of $\Gamma, \Pi, \alpha[i \leftarrow y] : \neg B$. Now either $y = \alpha(i)$, in which case $\alpha[i \leftarrow y] = \alpha$ and occurs in Γ, Π, or $\alpha[i \leftarrow y]$ must occur in Π, as the side condition $?(\alpha[i \leftarrow y])$ was satisfied for the last inference in the second derivation. In both cases we can apply IH to obtain a derivation of $\Gamma, \overline{W}\alpha, \alpha(i)\overline{R}_i y, \Gamma, \Pi$, and applications of (contraction), (R-contraction) and (W-contraction) complete the proof in this case. $\Box$

6 Completeness

In the cases where an axiomatisation of the (relativised) product is known, this can be used to show the completeness of the product calculi. As (cut) is admissible in the product calculi $\mathbf{LS}^{n,n}_{W\times}(T)$, the set of derivable formulas is closed under modus ponens. Obviously, it is also closed under necessitation. So completeness follows, if the axioms are derivable. In particular, for product-matching logics, the derivability of the commutator axioms yields completeness. A similar remark applies to the cases where the expanding commutator and the e-commutator coincide, and to the cases where the arbitrarily relativised product is indeed the fusion of the logics. This argument, however, does not cover, for example, the two most simple systems $\mathbf{LS}^{n,n}_{\times}(\emptyset)$ and $\mathbf{LS}^{n,0}_{\times}(\emptyset)$ which are candidates for the logics $\mathbf{K}^n$ and $\mathbf{S5}^n$. Hence we conclude with presenting a general, direct proof of completeness in the style of [15,14]. In the sequel, we sketch the argument and transfer more details to the appendix.

We consider a possibly infinite proof-search tree $\mathfrak{T}$ for a sequent Γ which is constructed by repeatedly extending a finite deduction tree at its leaves.

In every step, a rule of the system is applied bottom-up. If the result $\mathfrak{T}$ of this process is a finite tree where all leaves carry axioms, we have found a derivation of Γ. The construction is organised in such a manner that we can define a countermodel in the remaining cases. If $\mathfrak{T}$ contains a non-axiom leaf where no reduction step is applicable, we choose such a node, and let $\pi = N_1, \ldots, N_p$ denote the path from the root to that leaf. Otherwise the tree $\mathfrak{T}$ is a finitely-branching, infinite tree. Then let $\pi = N_1, \ldots, N_\nu, \ldots$ be an infinite maximal path starting at the root. Let Δ be the union of all judgements at the nodes N_ν of π. Define $\mathcal{W}_i := \{x \in L_i \mid x \text{ occurs in } \Delta\}$ for $i \in 1..n$, $\mathcal{R}_i := \{(x,y) \in \mathcal{W}_i \mid x\overline{R}_i y \text{ occurs in } \Delta\}$ for $i = 1..r$, $\mathcal{R}_i := \mathcal{W}_i \times \mathcal{W}_i$ for $i \in (r+1)..n$, and $\mathcal{F}_i := (\mathcal{W}_i, \mathcal{R}_i)$ for $i \in 1..n$. In case of $\mathbf{LS}^{n,r}_{W\times}(T)$, we let $\mathcal{W} := \{\alpha \in L \mid \alpha \text{ occurs in } \Delta\}$ otherwise $\mathcal{W} := \mathcal{W}_1 \times \ldots \times \mathcal{W}_n$. In case of the non-relativised product, we let $\mathcal{F}$ denote the product frame of the $\mathcal{F}_i$, otherwise the subframe of the product determined by $\mathcal{W}$. Define $\mathcal{V} : \mathcal{W} \to \mathrm{PV} \to \mathbb{B}$ by

$$\mathcal{V}(\alpha)(P) := \begin{cases} \mathbf{t} \text{ if } \alpha : \neg P \text{ occurs on } \pi \\ \mathbf{f} \text{ otherwise} \end{cases}$$

Note that $\mathcal{V}(\alpha)(P) = \mathbf{f}$ if $\alpha : P$ occurs on π, as $\alpha : P$ and $\alpha : \neg P$ can not occur both on π. Now consider the model $\mathcal{M} := (\mathcal{F}, \mathcal{V})$, let $\ell_i(x) = x$ for all $x \in \mathcal{W}_i$ and $\bar{\ell} = \ell_1, \ldots, \ell_n$. (For y not in π, $\ell_i(y)$ is irrelevant.) For this model, $(\mathcal{M}, \bar{\ell}) \not\models \alpha : A$ for all $\alpha : A$ in Δ can be shown by induction on A. This completes the proof of the following theorem:

Theorem 6.1 *Let A be a formula which is valid in all (subframes of) frames that are products of frames for C_i. Then $\boldsymbol{LS}^{n,r}_{(W)\times}(T) \vdash \alpha : A$.*

7 Conclusion

We have developed a general strategy for building *products* of labelled calculi. This way we obtain systems for a large class of products and relativised products of modal logics. In particular, we obtain rather handy systems for $\mathbf{K}^n$ and $\mathbf{S5}^n$. For product matching logics, these calculi offer an alternative to the axiomatisation as commutators. Note that the regular rule scheme is sufficient for the product if it is sufficient for the components. In contrast to the one-dimensional system obtained by transforming the frame conditions for the commutator into rules, the construction of the product calculus does not introduce rules with eigenvariables. We considered arbitrary relativisations, expanding relativisations and non-relativised products, and we presented proof-theoretic arguments for some basic facts. The theory of combinations of modal logics, however, comprises many results that were not touched here. Product calculi could be used to add proof-theoretic arguments to the picture but this is still up to further work.

Appendix

A Derivations of the commutator axioms

Derivation of $\mathrm{chr}_{i,j}$ in $\mathbf{LS}^{n,r}_{W\times}(T^{N\text{-EX}})$: Let $\Delta := \overline{W}\alpha, \alpha(i)\overline{R}_i x, \overline{W}\alpha, \alpha(j)\overline{R}_j y$ and $\Delta' := \Delta, \overline{W}\alpha[i,j\leftarrow x,y], \alpha[i\leftarrow x] : \Diamond_j\neg P$.

$$(\vee)\,\dfrac{(\Box_i)\,\dfrac{(\Box_j)\,\dfrac{(i\mathrm{EX})\,\dfrac{(\Diamond_j)\,\dfrac{(\Diamond_i)\,\dfrac{\Delta', \alpha[j\leftarrow y] : \Diamond_i P, \alpha[i,j\leftarrow x,y] : \neg P, \alpha[i,j\leftarrow x,y] : P}{\Delta, \overline{W}\alpha[i,j\leftarrow x,y], \alpha[i\leftarrow x] : \Diamond_j\neg P, \alpha[j\leftarrow y] : \Diamond_i P, \alpha[i,j\leftarrow x,y] : \neg P}}{\Delta, \overline{W}\alpha[i,j\leftarrow x,y], \alpha[i\leftarrow x] : \Diamond_j\neg P, \alpha[j\leftarrow y] : \Diamond_i P}}{\Delta, \alpha[i\leftarrow x] : \Diamond_j\neg P, \alpha[j\leftarrow y] : \Diamond_i P}}{\overline{W}\alpha, \alpha(i)\overline{R}_i x, \alpha[i\leftarrow x] : \Diamond_j\neg P, \alpha : \Box_j\Diamond_i P}}{\alpha : \Box_i\Diamond_j\neg P, \alpha : \Box_j\Diamond_i P}}{\alpha : \Box_i\Diamond_j\neg P \vee \Box_j\Diamond_i P}$$

Derivation of $\mathrm{com}^l_{i,j}$ in $\mathbf{LS}^{n,r}_{\times}(T)$ if $i,j \in 1..r$, $i \neq j$ where $x \in L_i$ and $y \in L_j$ are fresh:

$$(\vee)\,\dfrac{(\Box_j)\,\dfrac{(\Box_i)\,\dfrac{(\Diamond_i)\,\dfrac{(\Diamond_j)\,\dfrac{\alpha(j)\overline{R}_j y, \alpha(i)\overline{R}_i x, \alpha[i,j\leftarrow x,y] : \neg P, \alpha : \Diamond_i\Diamond_j P, \alpha[i\leftarrow x] : \Diamond_j P, \alpha[i,j\leftarrow x,y] : P}{\alpha(j)\overline{R}_j y, \alpha(i)\overline{R}_i x, \alpha[i,j\leftarrow x,y] : \neg P, \alpha : \Diamond_i\Diamond_j P, \alpha[i\leftarrow x] : \Diamond_j P}}{\alpha(j)\overline{R}_j y, \alpha(i)\overline{R}_i x, \alpha : \Diamond_i\Diamond_j P, \alpha[i,j\leftarrow x,y] : \neg P}}{\alpha(j)\overline{R}_j y, \alpha : \Diamond_i\Diamond_j P, \alpha[j\leftarrow y] : \Box_i\neg P}}{\alpha : \Box_j\Box_i\neg P, \alpha : \Diamond_i\Diamond_j P}}{\alpha : \Box_j\Box_i\neg P \vee \Diamond_i\Diamond_j P}$$

Derivation of $\mathrm{chr}_{i,j}$ in $\mathbf{LS}^{n,r}_{\times}(T)$ if $i,j \in 1..r$, $i \neq j$ where $x \in L_i$ and $y \in L_j$ are fresh:

$$(\vee)\,\dfrac{(\Box_i)\,\dfrac{(\Box_j)\,\dfrac{(\Diamond_j)\,\dfrac{(\Diamond_i)\,\dfrac{\alpha(i)\overline{R}_i x, \alpha(j)\overline{R}_j y, \alpha[i\leftarrow x] : \Diamond_j\neg P, \alpha[j\leftarrow y] : \Diamond_i P, \alpha[i,j\leftarrow x,y] : \neg P, \alpha[i,j\leftarrow x,y] : P}{\alpha(i)\overline{R}_i x, \alpha(j)\overline{R}_j y, \alpha[i\leftarrow x] : \Diamond_j\neg P, \alpha[j\leftarrow y] : \Diamond_i P, \alpha[i,j\leftarrow x,y] : \neg P}}{\alpha(i)\overline{R}_i x, \alpha(j)\overline{R}_j y, \alpha[i\leftarrow x] : \Diamond_j\neg P, \alpha[j\leftarrow y] : \Diamond_i P}}{\alpha(i)\overline{R}_i x, \alpha[i\leftarrow x] : \Diamond_j\neg P, \alpha : \Box_j\Diamond_i P}}{\alpha : \Box_i\Diamond_j\neg P, \alpha : \Box_j\Diamond_i P}}{\alpha : \Box_i\Diamond_j\neg P \vee \Box_j\Diamond_i P}$$

B Proof of Completeness

We construct a possibly infinite proof-search tree $\mathfrak{T}$ for a sequent Γ by repeatedly extending a finite deduction tree at its leaves. In every step, a rule of the system is applied bottom-up. First, we add some details concerning the single steps:

(i) In case of $(\vee)$ or $(\wedge)$, the expansion is completely determined. Note that applying $(\wedge)$ leads to a branching of the tree, duplicating the remaining applicable steps at this node. We can improve the procedure by observing that a reduction of $\alpha : A \wedge B$ is redundant if $\alpha : A$ or $\alpha : B$ already occurs on the path to the current node, and exclude redundant reductions. A similar remark applies to $\alpha : A \vee B$ if both $\alpha : A$ and $\alpha : B$ occur on the path to the node.

(ii) On reducing a $\Box_i$-formula, we have to choose an eigenvariable. We stipulate that this should be a label which is fresh with respect to the construction so far, and assume some arbitrary but fixed function which gives us the next fresh label of the requested type. A reduction of $\alpha : \Box_i A$ is redundant if some $\alpha[i \leftarrow x] : A$ does already occur on the path to the current node.

(iii) In the premiss of $(\Diamond_i)$, the label x which is substituted is not uniquely determined. Here we decide that, if $(\Diamond_i)$ is chosen, a finite sequence of $(\Diamond_i)$-steps should be performed, corresponding to different choices of x:

(a) In the calculus $\mathbf{LS}^{n,r}_{W\times}(T)$, the side condition $?(\alpha[i \leftarrow x])$ takes care that there are only finitely many alternatives. We postulate that all of them should be considered.

(b) In the calculus $\mathbf{LS}^{n,r}_{\times}(T)$, the label x could be any element of L_i. We decide that every label x which occurs on the path to the actual node should be considered. As the principal formula $\alpha : \Diamond_i A$ is repeated in the premiss, it stays reducible and will be reconsidered, as soon as new labels have been introduced.

We apply all those corresponding steps, for which the formula $\alpha[i \leftarrow x]$ does not already occur on the path to the node.

(iv) Relational rules are treated similarly. In general they may combine the aspects discussed above: branching of the tree, choosing an eigenvariable, considering several instances at once. These should be dealt with as in the case of logical rules. A bottom-up-application of a relational rule is redundant if, for some of its premisses, all atoms which would be introduced by this step are already present on the path to the current node.

We work on all leaves in parallel. The process is nevertheless non-deterministic, as we do not fix how to choose the steps. We just give some rules. Observe that, if a possible reduction is not chosen, the relevant formulas for this step are copied to all premisses. We call these copies and the copies thereof upwards on the path *descendants*. The postulates are as follows:

Termination: Nodes that carry axioms are not extended.

Continuation: If a node does not carry an axiom and at least one non-redundant step is possible, then one of these steps is chosen and performed.

Fairness: For all nodes N, non-redundant reduction steps at this node and the corresponding relevant formulas, as well as paths π in the final tree starting at N, we have the following: If none of the descendants of the relevant formulas are chosen on that path, although the corresponding reduction stays non-redundant, then the path ends with a leaf carrying an axiom.

Fairness can be achieved by assigning at every stage higher priority to those reductions already enabled to those constructed by choosing an alternative step.

Now consider the result $\mathfrak{T}$ of this process. If this a finite tree where all leaves carry axioms, we have found a derivation of Γ. If it contains a non-axiom leaf where no reduction step is applicable, we choose such a node, and let $\pi = N_1, \ldots, N_p$ denote the path from the root to that leaf. If none of these is true, then the tree $\mathfrak{T}$ is a finitely-branching, infinite tree. Then let $\pi = N_1, \ldots, N_\nu, \ldots$ be an infinite maximal path starting at the root. Let Δ be the union of all judgements at the nodes N_ν. Based on Δ, we define a countermodel.

Let $\mathcal{W}_i := \{x \in L_i \mid x \text{ occurs in } \Delta\}$ for $i \in 1..n$, $\mathcal{R}_i := \{(x,y) \in \mathcal{W}_i \mid x\overline{R}_i y \text{ occurs in } \Delta\}$ for $i = 1..r$, $\mathcal{R}_i := \mathcal{W}_i \times \mathcal{W}_i$ for $i \in (r+1)..n$, and $\mathcal{F}_i := (\mathcal{W}_i, \mathcal{R}_i)$ for $i \in 1..n$. In case of $\mathbf{LS}^{n,r}_{W\times}(T)$, let $\mathcal{W} := \{\alpha \in L \mid \alpha \text{ occurs in } \Delta\}$ otherwise $\mathcal{W} := \mathcal{W}_1 \times \ldots \times \mathcal{W}_n$. Observe that the conditions given by the rules in T are satisfied. To see this, consider a frame condition

$$\forall \bar{y} \forall \bar{z} (\bigwedge_{\nu=1}^{k} U_\nu \wedge W\alpha_1 \wedge \ldots \wedge W\alpha_\ell \rightarrow \exists \bar{x} \bigvee_{\mu=1}^{m} \bigwedge_{\rho=1}^{l_m} V^\mu_\nu)$$

and the corresponding frame rule

$$\frac{\Gamma, U_1, \ldots, U_k, V^1_1, \ldots, V^1_{l_1} \qquad \ldots \qquad \Gamma, U_1, \ldots, U_k, V^m_1, \ldots, V^m_{l_m}}{\Gamma, U_1, \ldots, U_k} \,!(\bar{x})?(\bar{\alpha})$$

Here $\bar{y}$ is the list of labels in $U_1, \ldots, U_k$ and $\bar{z}$ is the list of labels in $V^1_1, \ldots, V^m_{l_m}$ that are neither eigenvariables nor occur in $\bar{y}$. (We assume $\bar{x}, \bar{y}, \bar{z}$ to be pairwise distinct.) Let θ be a finite label substitution, replacing the labels $\bar{y}, \bar{z}$ by appropriate labels in $\bigcup_{i=1}^{n} \mathcal{W}_i$ so that its application turns $\forall \bar{y} \forall \bar{z} (\bigwedge_{\nu=1}^{k} U_\nu \wedge W\alpha_1 \wedge \ldots \wedge W\alpha_\ell)$ into a true assertion where we assume the interpretation given by $(\mathcal{W}, \mathcal{R}_1, \ldots, \mathcal{R}_n)$. This means that $\alpha_1\theta, \ldots, \alpha_\ell\theta$ as well as $U_1\theta, \ldots, U_k\theta$ occur at some nodes of π. Furthermore, the labels $z_\rho\theta$ occur in Δ. As relational atoms in the conclusion of an inference are also contained in all premisses, there must be some node N of π where *all* these atoms are present and all $z_\rho\theta$ occur on the path from the root to N. At this point, the application of the relational rule would be possible. As π does not contain an axiom, the reduction of the descendants must either become redundant or actually be performed. In both cases, for some $\mu \in 1..m$, all atoms $V^\mu_1\theta, \ldots, V^\mu_{l_\mu}\theta$ occur on the path π. The verification of the frame conditions of C_i is similar but simpler. Consequently, the frames $(\mathcal{W}_i, \mathcal{R}_i)$ are frames for C_i. In case of the non-relativised product, we let $\mathcal{F}$ denote their product frame, otherwise the subframe of the product determined by $\mathcal{W}$. We define $\mathcal{V} : \mathcal{W} \rightarrow \mathrm{PV} \rightarrow \mathbb{B}$ by

$$\mathcal{V}(\alpha)(P) := \begin{cases} \mathbf{t} \text{ if } \alpha : \neg P \text{ occurs on } \pi \\ \mathbf{f} \text{ otherwise} \end{cases}$$

Note that $\mathcal{V}(\alpha)(P) = \mathbf{f}$ if $\alpha : P$ occurs on π, as $\alpha : P$ and $\alpha : \neg P$ can not occur both on π.

We consider the model $\mathcal{M} := (\mathcal{F}, \mathcal{V})$, let $\ell_i(x) = x$ for all $x \in \mathcal{W}_i$, and $\bar{\ell} = \ell_1, \ldots, \ell_n$. (For y not in π, $\ell_i(y)$ is irrelevant.) For this model, $(\mathcal{M}, \bar{\ell}) \not\models \alpha : A$ for all $\alpha : A$ in Δ , which completes the proof of the theorem, and can be shown by induction on A. We present the case $A \equiv \Diamond_i B$ in the version with $\overline{W}$. Let $x \in L_i$ so that $\alpha[i \leftarrow x]$ and $\alpha(i)\overline{R}_i x$ occur on π. Then at some node N of π, there is a sequent Γ' containing $\alpha : \Diamond_i B$ so that $\alpha[i \leftarrow x]$ and $\alpha(i)\overline{R}_i x$ occur in Γ'. Then either the corresponding reduction is performed on the path or it becomes redundant. In both cases, we know that $\alpha[i \leftarrow x] : B$ occurs on π. Application of the IH yields $(\mathcal{M}, \bar{\ell}) \not\models \alpha[i \leftarrow x] : B$. The remaining cases are similar.

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